8

Process experimentation with two or more factors

Many important phenomena depend, not on the operation of a single factor, but on the bringing together of two or sometimes more factors at the appropriate levels. (Box, 1990, p. 365)

Overview

The design of experiments (DOE) is a vast topic – designed experiments are major tools for use in the improve phase of many Six Sigma projects. In the previous chapter experiments involving only a single factor of interest were considered. In this chapter experiments involving two or more factors of interest will be introduced. Much process improvement experimentation is done on a one-factor-at-a-time basis. Although such experimentation can lead to improvement, there is no doubt that multifactor experiments, in which factor levels are varied systematically according to a recognized design, can be much more informative. In particular, multifactor experiments can reveal the presence of important interactions between factors. Harnessing interaction effects can lead to dramatic process improvements.

Minitab can be of assistance both with the actual design of the experiment and with the display and analysis of the resulting data. Following a general introduction, in which the concept of interaction will be introduced, experiments in the 2^k series, with k factors, each with two levels, will be considered. Screening experiments and fractional factorial experiments in the 2^{k-p} series will be introduced, together with the concept of design resolution. The fundamentals of response surfaces will be described. Reference will be made to Taguchi experimental designs.

8.1   General factorial experiments

8.1.1   Creation of a general factorial experimental design

As an introductory example, consider the process of making popcorn. Two factors are of interest – the first is the type of popper (X_1) and the second is the grade of corn used (X_2). The levels of interest for popper are air and oil, those for corn are budget, regular and luxury. The response of interest is the volume (ml) yield of popcorn (Y) from 250 ml of corn processed according to the instructions provided by the manufacturers of the machines. The design may be created using **Stat** > **DOE** > **Factorial** > **Create Factorial Design. . .** for which the initial dialog is shown in Figure 8.1.

The first step is to select **General full factorial design** with **Number of factors:** 2 specified. (With the generality of this type of design it is not possible to provide a catalogue of designs under **Display Available Designs. . .**.) The second step is to click on **Designs. . .** and engage in the subdialog as indicated in Figure 8.2.

The name of each factor may be entered together with the number of levels – in this case popper has two levels and corn has three. The default **Number of Replicates:** is 1, but in this experiment 4 was the number used. This means that the process was operated four times with each of the 2 (levels of popper) $\times$ 3 (levels of corn) $= 6$ factor-level combinations (FLCs) – we say that it was *replicated* four times. Thus the term 'replicate' is being used in a technical statistical sense in the context of designed experimentation. (When more than one replicate is employed the opportunity to block on replicates is made available but was not required in this case.) Having clicked **OK**, click on **Factors. . .** and complete the dialog shown in Figure 8.3. Here the categorical levels for the factors mean that **Type** should be selected as Text. (Numeric would be selected in the case of a factor such as temperature specified in degrees Celsius.)

Under **Options. . .** the default to **Randomize runs** is strongly advised as part of good experimental design practice. (If the user enters an integer under **Base for random data generator:** then subsequent use of **Randomize runs** with **Create Factorial Designs** and the
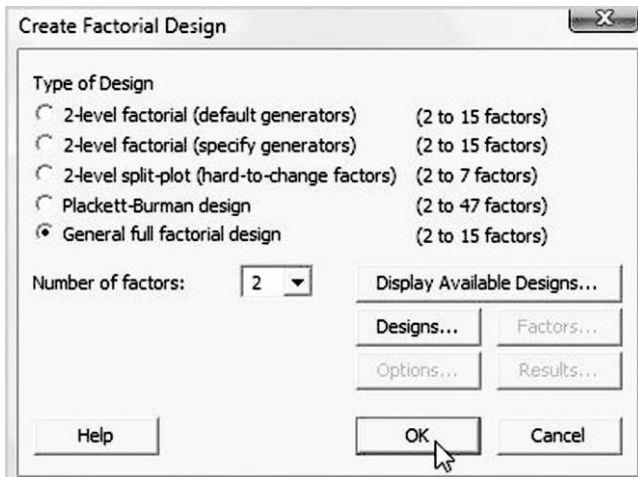

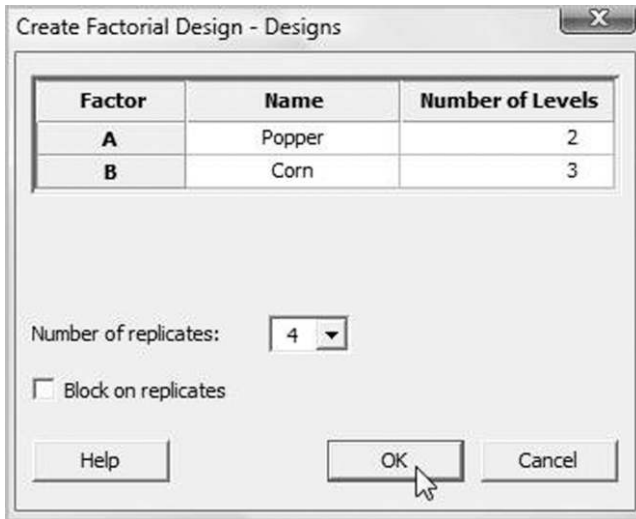**Figure 8.1**   Initial dialog for general full factorial design.

Figure 8.2 Specifying the factors.

same numbers of factors and levels and the same base leads to the same run order.) Opting to **Store design in worksheet**, which should of course be saved, is essential if data from the experiment are to be readily analysed later.

Having clicked **OK** and then **OK** again the design worksheet is created. The version obtained by the author is shown in Figure 8.4. It was augmented through the addition of a column named Yield for the recording of the yield obtained from each run and a column named Remarks in which those carrying out the experiment may note any unusual occurrences etc. It also has a column of residuals and a column of fitted values obtained during analysis of the data following completion of the experiment.

The columns created by Minitab are as follows:

- a column showing the standard order for each run – the reader is invited to create the worksheet again with the **Randomize runs** option unchecked and to scrutinize the systematic pattern in the standard order column;

- a column showing the actual run order created by the randomization procedure;
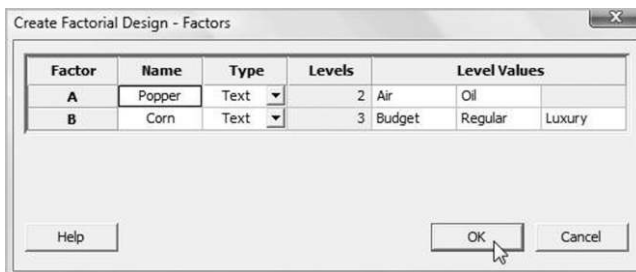

Figure 8.3 Specifying the factors: name, type, number of levels and level values.

C1	C2	C3	C4	C5-T	C6-T	C7	C8-T	C9	C10
StdOrder	RunOrder	PtType	Blocks	Popper	Corn	Yield	Remarks	RESI1	FITS1
9	1	1	1	Air	Luxury	1360		-112.125	1472.12
12	2	1	1	Oil	Luxury	1349		52.625	1296.37
10	3	1	1	Oil	Budget	943	Much lower yield from Budget	-27.625	970.62
14	4	1	1	Air	Regular	1090		-199.250	1289.25
6	5	1	1	Oil	Luxury	1108		-188.375	1296.37
21	6	1	1	Air	Luxury	1518		45.875	1472.12
8	7	1	1	Air	Regular	1342		52.750	1289.25
16	8	1	1	Oil	Budget	948		-22.625	970.62
23	9	1	1	Oil	Regular	1289		175.500	1113.50
7	10	1	1	Air	Budget	1120		-26.375	1146.37
24	11	1	1	Oil	Luxury	1241		-55.375	1296.37
2	12	1	1	Air	Regular	1537		247.750	1289.25
13	13	1	1	Air	Budget	1106		-40.375	1146.37
11	14	1	1	Oil	Regular	1148		34.500	1113.50
17	15	1	1	Oil	Regular	1058		-55.500	1113.50
4	16	1	1	Oil	Budget	923		-47.625	970.62
22	17	1	1	Oil	Budget	1164		193.375	970.62
1	18	1	1	Air	Budget	1198		51.625	1146.37
19	19	1	1	Air	Budget	1066		-80.375	1146.37
5	20	1	1	Oil	Regular	887		-226.500	1113.50
3	21	1	1	Air	Luxury	1585		112.875	1472.12
15	22	1	1	Air	Luxury	1449		-23.125	1472.12
20	23	1	1	Air	Regular	1260		-29.250	1289.25
18	24	1	1	Oil	Luxury	1464	Phew!	167.625	1296.37

Figure 8.4 Design worksheet with yields, remarks, residuals and fits from additive model.

- a column indicating point type in what is known as the design space – an explanation will be given later in the chapter (here all points are of type 1);

- a column indicating block – when there is no actual blocking this column consists entirely of 1s.

Armed with a copy of the worksheet, with the additional columns for yield and remarks added, the experimenters could head for the kitchen. The first run would have been with the air popper using 250 ml of luxury grade corn and would have given a yield of 1360 ml of popped corn, the second run would have been with the oil popper using 250 ml of luxury grade corn and would have given a yield of 1349 ml of popped corn and so on. Once the experimental work is complete display and analysis of the data can begin. The first eight columns displayed in Figure 8.4 are provided in worksheet Popcorn.MTW. The columns of residuals and fitted values in Figure 8.4 will be referred to later in the chapter.

8.1.2 Display and analysis of data from a general factorial experiment

The data may be presented in the form of a two-way table (Table 8.1) with rows corresponding to the levels of the popper factor and with columns corresponding to the levels of the corn factor. Each cell in the table gives the four replicate yields obtained with one particular FLC for the factors popper and corn. The mean yield for each cell is displayed in Table 8.2 together with the means corresponding to the rows (levels of popper) and columns (levels of corn) and the overall mean yield for the experiment.

Two forms of data display of these means are invaluable when dealing with data from experiments involving two or more factors – main effects plots and interaction plots. Use **Stat > ANOVA > Main Effects Plot...** to create the main effects plot; enter Yield under **Reponses:** and Popper and Corn under **Factors:**. The plot is shown in Figure 8.5.

The horizontal reference line indicates the overall mean yield for the experiment of 1214.7. The mean yields for each of the two levels of popper are plotted in the first panel and the mean yields for each of the three levels of corn are plotted in the second. Two insights are obtained. The first is that, on average, switching from the air popper to the oil popper reduces yield by

Table 8.1   Raw data for popcorn experiment.

Yield	Budget				Regular				Luxury			
Air	1120				1090				1360			
		1066				1342				1518		
			1106				1537				1585	
				1198				1260				1449
Oil	943				1289				1349			
		948				887				1108		
			923				1148				1241	
				1164				1058				1464

Table 8.2   Means summary of yield.

Mean Yield	Budget	Regular	Luxury	Mean
Air	1122.5	1307.3	1478.0	1302.6
Oil	994.5	1095.5	1290.5	1126.8
Mean	1058.5	1201.4	1384.3	1214.7

about 200 ml. The second is that, on average, yield progressively increases as we switch from budget grade corn to regular to luxury.

Use **Stat > ANOVA > Interactions Plot…** to create the interactions plot; enter Yield under **Reponses:** and Popper and Corn under **Factors:**. The plot is shown in Figure 8.6.

In this plot the six means at the core (shaded) of Table 8.2, each of which corresponds to an FLC, are displayed. The levels of corn are indicated on the horizontal axis. The levels of popper (see the legend in the top right-hand corner of the display) are indicated through the connection by line segments of those points that have the same level of popper. The pairs of line segments
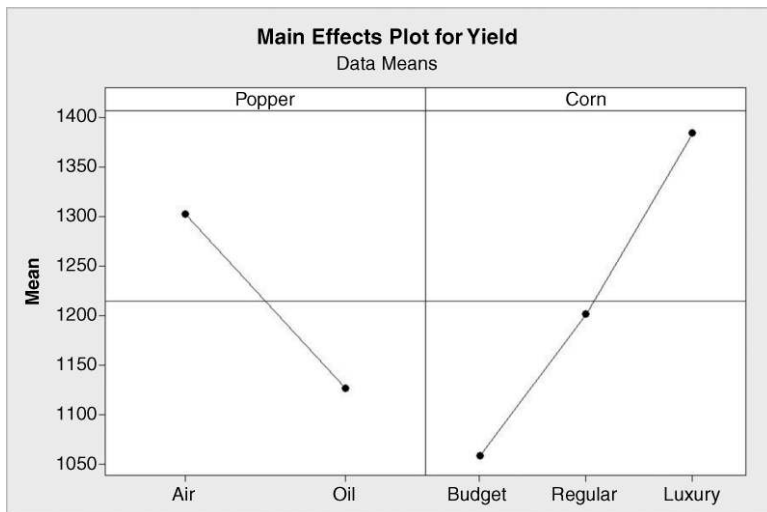

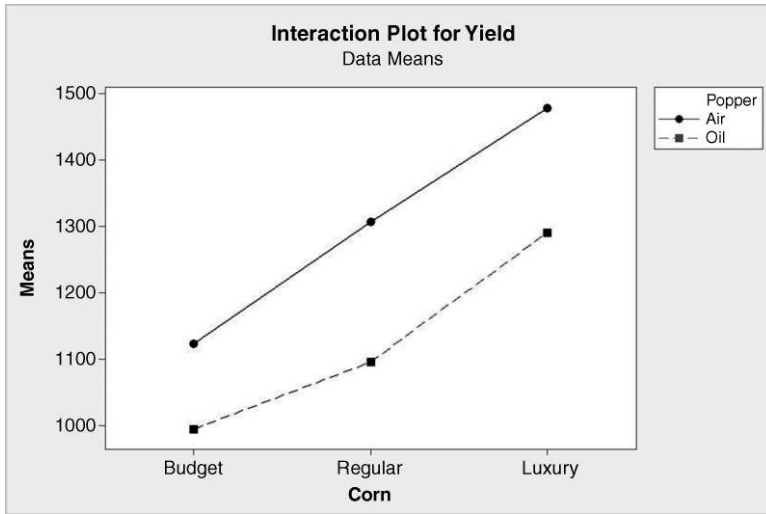**Figure 8.5**   Main effects plot for popcorn experiment.

Figure 8.6   Interaction plot – first version.

lying vertically above each other are approximately parallel. This means that, when using one particular grade of corn, similar decreases in mean yield are experienced on changing from the level air to the level oil of popper. This is characteristic of a situation in which there is no interaction between factors.

To create an alternative version of the interactions plot enter Yield under **Reponses:** and Corn and Popper under **Factors:** i.e. reverse the order in which the factors are entered. The plot is shown in Figure 8.7. The levels of popper are indicated on the horizontal axis. The levels of the factor corn are indicated through connection by line segments of those points that have the same level of corn, as indicated by the legend. Again the segments lying vertically above and
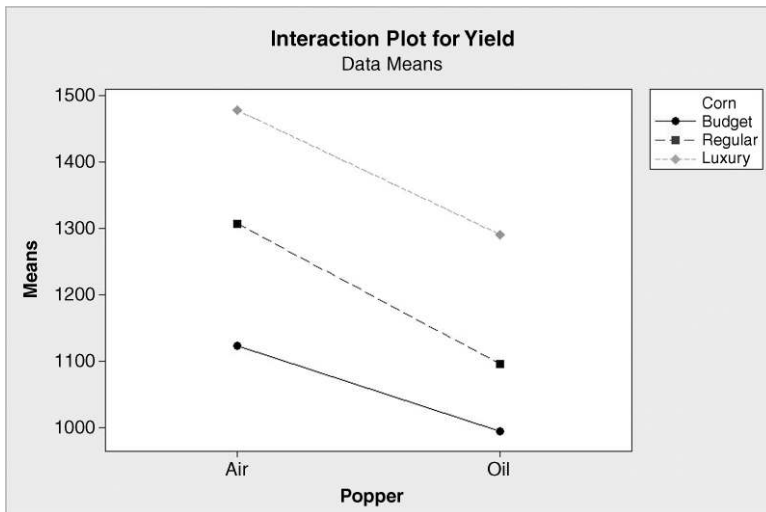

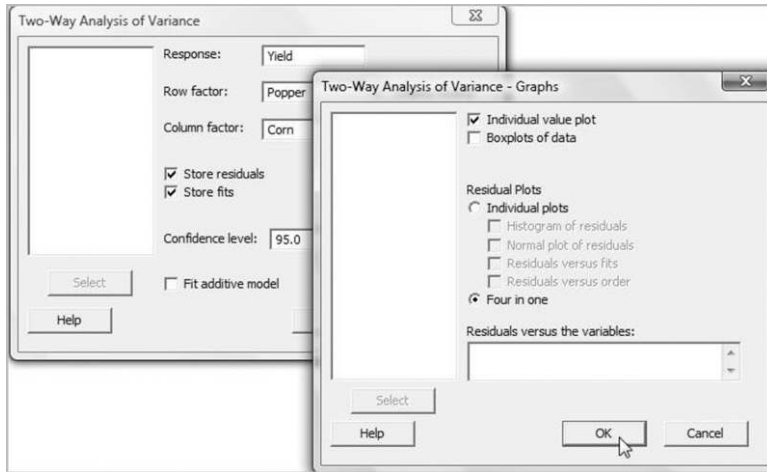**Figure 8.7**   Interaction plot – second version.

Figure 8.8   Dialog for two-way ANOVA.

below each other are approximately parallel. This means that, on changing the level of corn, similar changes in mean yield will be experienced, whether one is using an air popper or an oil popper.

If desired, both versions of the interaction plot may be created simultaneously by checking **Display full interaction plot matrix** in the dialog.

The data may be examined formally for evidence of interaction between the factors popper and corn by performing a two-way analysis of variance via **Stat > ANOVA > Two-Way....**. The terminology 'two-way' relates to the presentation of data from a general factorial experiment involving two factors in the form of a two-way table, as in Table 8.1. The dialog is shown in Figure 8.8.

Note carefully that the **Fit additive model** option must not be checked as we wish to formally assess the evidence for the presence of interaction. Under **Graphs...** it is strongly recommended that either the **Individual value plot** or **Boxplots of data** display be selected – when the number of replications is small the author recommends that the first of these plots be used. The **Four in one** option for **Residual Plots** is appropriate in this case since the observations of yield in the experiment are recorded in the worksheet in the time order in which they were obtained. The individual value plot is shown in Figure 8.9.

Note that the individual value plot in Figure 8.9 may be viewed as an 'exploded' version of the interactions plot in Figure 8.6 in which the individual yield values are plotted in addition to the yield means. The reader is invited to change the ordering of the factors in the dialog displayed on Figure 8.8 and to compare the individual value plot that is obtained with the interactions plot displayed in Figure 8.7.

As with the main effects plot in Figure 8.5 and the interaction plots in Figures 8.6 and 8.7, the plot in Figure 8.9 gives similar insights. It appears that:

- on average, switching from the air popper to the oil popper reduces yield by about 200 ml;

- on average, yield progressively increases as we switch from budget grade corn to regular to luxury.
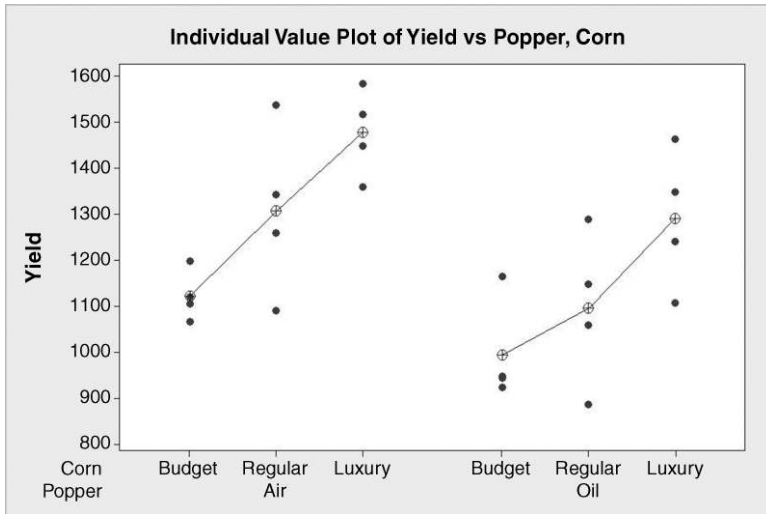
Figure 8.9   Individual value plot of yield data.

The standard residual plots are shown in Figure 8.10. The normal probability plot of the residuals is reasonably straight. Thus the assumption of a normal distribution of yield, for each combination of factor levels, is supported. Support for the assumption of a common variance for the distributions of yield for the six FLCs is provided by the similar vertical spread in all six groups of points in the plot of residuals versus fitted values. We can therefore proceed to interpret the ANOVA table with confidence that the assumptions underlying the valid application of the F-tests are reasonable.
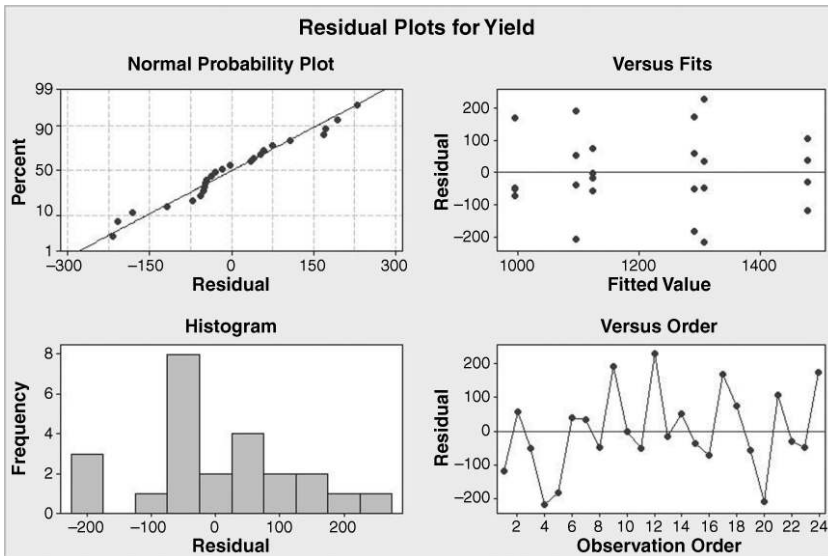

Figure 8.10   Residual plots.

```
Two-way ANOVA: Yield versus Corn, Popper

Source        DF      SS      MS      F      P
Corn           2  426586  213293  11.51  0.001
Popper         1  185328  185328  10.01  0.005
Interaction    2    7428    3714   0.20  0.820
Error         18  333423   18523
Total         23  952765

S = 136.1   R-Sq = 65.00%   R-Sq(adj) = 55.28%
```

Panel 8.1 ANOVA for model with interaction.

The ANOVA table is shown in Panel 8.1. In total we have a sample of 24 values of yield from the experiment so there are $24 - 1 = 23$ degrees of freedom (DF) in total. With 2 levels of popper there is $2 - 1 = 1$ degree of freedom for popper, and similarly $3 - 1 = 2$ degrees of freedom for corn. The number of degrees of freedom for the interaction between popper and corn is the product of the numbers of degrees of freedom for these two factors, i.e. $1 \times 2 = 2$. Thus the number of degrees of freedom for error is obtained by calculating $23 - (1 + 2 + 2) = 18$. The sums of squares (SS) are calculated using formulae that need not concern us; the mean squares (MS) are obtained by dividing each SS by its corresponding DF. Provided the assumptions referred to above are valid, the ratios of the MS values for popper, corn and interaction will have F-distributions. These enable the calculation of P-values.

The experiment provides no evidence of interaction between popper and corn since the P-value for interaction of 0.820 is well in excess of 0.05. This means that there is no evidence from the experiment that the effect on yield of changing the grade of corn used depends on the type of popper being used. When there is no evidence of interaction in a two-factor experiment of this nature one can proceed to fit an additive model. This may readily be achieved using **Edit Last Dialog** and checking **Fit additive model**. In addition **Store residuals** and **Store fits** were checked in order that fitted values and residuals were available for explanation that follows. These fitted values and residuals are displayed in the columns labelled RESI1 and FITS1 in Figure 8.4. The residual plots were again satisfactory (but are not displayed here) and the revised ANOVA table is shown Panel 8.2.

For popper the P-value is 0.004 so this means that the main effect of popper is significant at the 1% level. The main effect of corn is significant at the 0.1% level. Thus the experiment provides strong evidence that the factors of interest – type of popper (X_1) and grade of corn used (X_2) – influence the response of interest, i.e. the volume (ml) yield of popcorn (Y), from 250 ml of corn processed according to the instructions provided by the manufacturers of the machines.

We will now introduce and fit the formal additive model underlying the analysis. Reference to Table 8.2 enables the effects for popper to be calculated as indicated in Table 8.3. The overall

```
Two-way ANOVA: Yield versus Corn, Popper

Source  DF      SS      MS      F      P
Corn     2  426586  213293  12.52  0.000
Popper   1  185328  185328  10.87  0.004
Error   20  340851   17043
Total   23  952765

S = 130.5   R-Sq = 64.23%   R-Sq(adj) = 58.86%
```

Panel 8.2 ANOVA for additive model.

Table 8.3   Effects for popper.

Level	Mean yield	Effect
Air	1302.6	87.9
Oil	1126.8	− 87.9

mean yield for the experiment was 1214.7 ml. The effect corresponding to a particular level of a factor is the mean yield for that factor level minus the overall mean yield for the experiment. Hence the air popper effect is $1302.6 − 1214.7 = 87.9$ and the oil popper effect is $1126.8 – 1214.7 = − 87.9$. Thus we can interpret the effect of using the air popper as being to elevate yield from the overall mean of 1214.7 by 87.9 ml on average, and the effect of using the oil popper as being to depress (because of the negative effect) yield by 87.9 ml on average. Note that the two effects sum to 0.

Reference to Table 8.2 enables the effects for corn to be calculated as indicated in Table 8.4, and the reader is invited to confirm the calculations. Thus we can interpret the effect of using the budget grade corn as being to depress yield from the overall mean of 1214.7 by 156.2 ml on average, that of using the regular grade being to depress yield by 13.3 ml on average and that of using the luxury grade being to elevate yield by 169.6 ml on average. Note that the three effects sum to 0 (allowing for rounding of the means quoted in Table 8.2 to one decimal place).

The reader will recall from the previous chapter that

$$\text{Observed data value} = \text{Value fitted by model} + \text{Residual,}$$

which we abbreviated as

$$\text{Data} = \text{Fit} + \text{Residual.}$$

In this case we take

$$\text{Fit} = \text{Overall mean} + \text{Popper effect} + \text{Corn effect.}$$

The first experimental run was carried out using an air popper with luxury grade corn and gave a yield of 1360 (see Figure 8.4). The fitted value is given by

$$\text{Fit} = \text{Overall mean} + \text{Air popper effect} + \text{Luxury grade of corn effect}$$
$$= 1214.7 + 87.9 + 169.6 = 1472.2.$$

Thus the corresponding residual may be obtained as

$$\text{Residual} = \text{Data} − \text{Fit} = 1360 − 1472.2 = − 112.2.$$

Table 8.4   Effects for corn.

	Corn		
Level	Budget	Regular	Luxury
Mean yield	1058.5	1201.4	1384.3
Effect	− 156.2	− 13.3	169.6

Scrutiny of the column of residuals computed by Minitab (see Figure 8.4) reveals a value of -112.125. Again the small discrepancy is due to rounding. The reader is invited to calculate the next few residuals in order to check his/her understanding.

8.1.3   The fixed effects model, comparisons

The *fixed* effects model for a general two-factor design, *with no interaction*, is specified in Box 8.1. The number of levels of the row factor is a, the number of levels of the column factor is b and the number of replications is n. For example, with $i = 2$, $j = 3$ and $k = 1$ we have the specific relationship

$$Y_{231} = \mu + \alpha_2 + \beta_3 + \varepsilon_{231}.$$

For the popcorn experiment this states that the yield from the first run ($k = 1$, indicating the first replicate) with the oil type of popper ($i = 2$, indicating the second level of the factor popper) and the luxury grade of corn ($j = 3$, indicating the third level of the factor corn) is made up of the overall mean plus the effect of oil type of popper plus the effect of luxury grade corn plus a random error. Earlier we estimated the overall mean, μ, as 1214.7, the effect of oil type popper, α_2, as -87.9 and the effect of luxury grade corn, β_3, as 169.6. Thus the fitted value for the combination of oil type of popper with luxury corn is the sum $1214.7 + (-87.9) + 169.6 = 1296.4$. The value $s = 130.5$, which is given beneath the ANOVA table and is the square root of the error MS in the ANOVA table in Panel 8.2, provides an estimate of the standard deviation σ of the random error component of the model. The model predicts that the population of yields obtained that would be obtained with the combination of oil type of popper and luxury grade corn would be normally distributed with mean 1296.4 and standard deviation 130.5. Similar statements may be made about the other FLCs.

Follow-up comparisons are available via **Stat > ANOVA > General Linear Model...**. The dialog is shown in Figure 8.11. In **Model:** we are communicating to Minitab the nature of the model we are using, i.e. $Y_{ijk} = \mu + \alpha_i + \beta_j + \varepsilon_{ijk}$. Such models always include an overall mean and the random error term, so by entering Popper and Corn in **Model:** we are indicating the $\alpha_i + \beta_j$ terms in the core of the model. In the **Comparisons...** subdialog, **Pairwise comparisons** were selected, by the **Tukey** method. **Grouping information** and **Confidence interval** were both checked, with the default percentage **Confidence level:** 95.0 used. Part of the corresponding section of the Session window output is shown in Panel 8.3.

Observed data value = Overall mean + Row factor effect + Column factor effect
$$+ \text{Random error}$$

$$Y_{ijk} = \mu + \alpha_i + \beta_j + \varepsilon_{ijk}, \quad i = 1, 2, \ldots, a, \quad j = 1, 2, \ldots, b, \quad k = 1, 2, \ldots, n$$

$$\sum_{i=1}^{a} \alpha_i = 0, \quad \sum_{j=1}^{b} \beta_j = 0, \quad \varepsilon_{ijk} \sim N(0, \sigma^2)$$

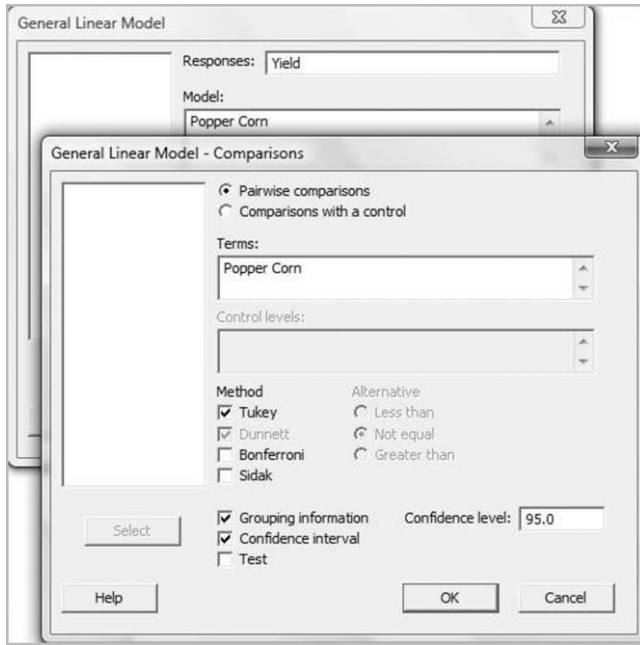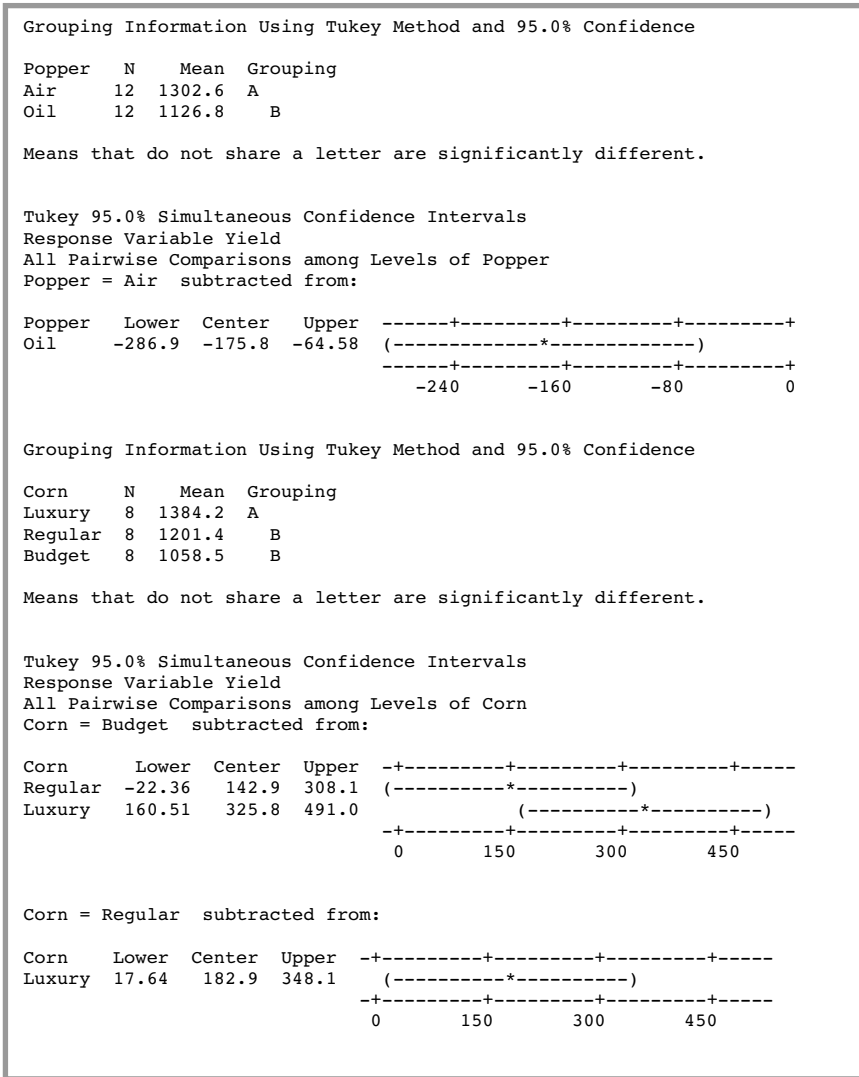**Box 8.1**   Additive fixed effects model for two factors.

Figure 8.11   General Linear Model dialog.

The interpretation, with rounding of values, is as follows:

- On average the yield with the oil type popper is 176 ml lower than that with the air type popper, with confidence interval 65 ml to 287 ml (to the nearest integer).

- On average the yield with regular grade corn is 143 ml higher than with budget grade corn, with confidence interval -22 to 308 ml (to the nearest integer).

- On average the yield with luxury grade corn is 326 ml higher than with budget grade corn, with confidence interval 161 to 491 ml (to the nearest integer).

- On average the yield with luxury grade corn is 183 ml higher than with regular grade corn, with confidence interval 18 to 348 ml (to the nearest integer).

Since the second confidence interval includes 0, we have no evidence that the mean yield with budget grade corn differs significantly from that with regular grade corn. The other confidence intervals provide evidence of the superiority in terms of yield of an air popper with luxury grade corn. The reader may, of course, arrive at the key conclusions by scrutinizing the grouping information provided in the Session window output prior to each set of confidence intervals.

If the aim is to maximize yield (Y) using a combination of the fixed set of levels of the factors popper (X_1) and corn grade (X_2) considered, the levels air and luxury should be selected. There are indications of a clear 'winning combination' in this scenario. Note that this combination corresponds to selection of the level for popper corresponding to the greatest mean for the levels of popper and to the selection of the level for corn corresponding to the

```
Grouping Information Using Tukey Method and 95.0% Confidence

Popper   N    Mean  Grouping
Air     12  1302.6  A
Oil     12  1126.8     B

Means that do not share a letter are significantly different.


Tukey 95.0% Simultaneous Confidence Intervals
Response Variable Yield
All Pairwise Comparisons among Levels of Popper
Popper = Air   subtracted from:

Popper  Lower  Center   Upper  ------+---------+---------+---------+
Oil    -286.9  -175.8  -64.58  (-------------*-------------)
                               ------+---------+---------+---------+
                                  -240       -160       -80        0


Grouping Information Using Tukey Method and 95.0% Confidence

Corn    N    Mean  Grouping
Luxury  8  1384.2  A
Regular 8  1201.4     B
Budget  8  1058.5     B

Means that do not share a letter are significantly different.


Tukey 95.0% Simultaneous Confidence Intervals
Response Variable Yield
All Pairwise Comparisons among Levels of Corn
Corn = Budget  subtracted from:

Corn      Lower  Center  Upper  -+---------+---------+---------+-----
Regular  -22.36   142.9  308.1  (----------*----------)
Luxury   160.51   325.8  491.0            (----------*----------)
                                -+---------+---------+---------+-----
                                 0       150       300       450


Corn = Regular  subtracted from:

Corn     Lower  Center  Upper  -+---------+---------+---------+-----
Luxury   17.64   182.9  348.1   (----------*----------)
                               -+---------+---------+---------+-----
                                0       150       300       450
```

Panel 8.3   Session window output of comparisons.

greatest mean for the levels of corn – see the main effects plot in Figure 8.5. However, it should be borne in mind that other responses might be used in decision-making, such as flavour and costs. It is also important to be aware that, when there are significant interaction effects, this pick-a-winner approach can lead to selection of an FLC that is not optimal.

As a second example, consider a hypothetical experiment performed by an internet retailer prior to the launch of new laptop computer on the market. The retailer manages its database of customers in homogeneous marketing groups of approximately 5000 customers. The factors of interest were price (X_1), with levels £499, £549 and £599, and offer (X_2), with levels software and wireless, and the response was number of sales (Y). Two replications were made of the full 3×2 factorial (the shorthand 3×2 indicating that the first factor has three levels and the second factor has two levels.) Thus each customer in two groups selected at random was sent an

Table 8.5   Data for marketing experiment.

Sales	Offer	
Price	Software	Wireless
£499	136	98
	140	110
£549	72	96
	74	114
£599	69	102
	61	92

e-mail inviting purchase of the new laptop for price £449 with the offer of free software. Customers in a second pair of groups selected at random received the offer of the new laptop for price £449 with the offer of a free wireless networking module, and so on. The data, displayed in Table 8.5, are available in the worksheet Marketing.MTW.

An individual values plot of the data is shown in Figure 8.12. The plot was obtained via **Stat > ANOVA > Two-Way...** and **Graphs...** with **Individual value plot** selected. **Fit additive model** was left unchecked and the ANOVA table shown in Panel 8.4 obtained.

There is evidence of interaction in this case, with P-value 0.001. In such cases emphasis is put on understanding the nature of the interaction rather than on interpretation of the main effects. Use was made of **Stat > ANOVA > Interactions Plot...** with the option **Display full interaction plot matrix** checked to create the interaction plots shown in Figure 8.13. Unlike the previous example the vertical bands of line segments are far from parallel in some cases. (In fact there is a crossing over in one case.) This is typical in situations where there is significant interaction between factors. There is evidence here that the effect on sales of changing the nature of the offer, from one of free software to one of a free wireless networking
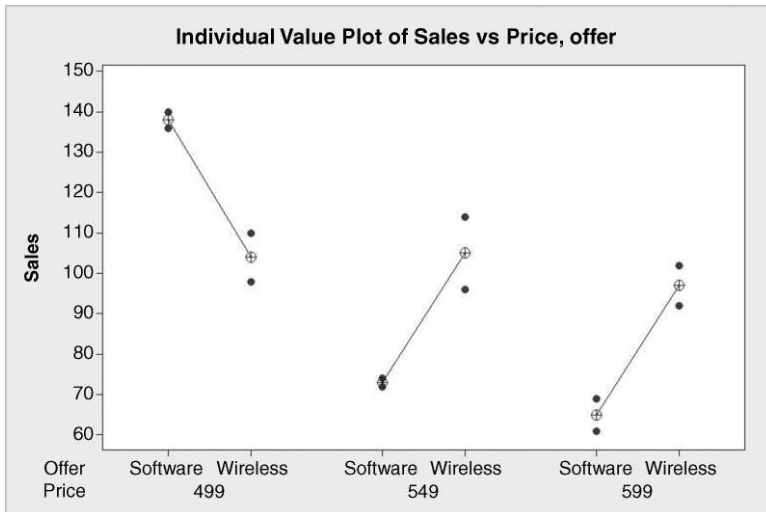

Figure 8.12   Individual value plot of marketing data.

```
Two-way ANOVA: Sales versus Price, Offer

Source        DF    SS      MS      F      P
Price          2  3584  1792.00  32.98  0.001
Offer          1   300   300.00   5.52  0.057
Interaction    2  2904  1452.00  26.72  0.001
Error          6   326    54.33
Total         11  7114

S = 7.371   R-Sq = 95.42%   R-Sq(adj) = 91.60%
```

Panel 8.4   ANOVA for marketing experiment.

module, depends on the level of price. For example, with price set at £499 there was a greater level of sales with the offer of software than with the offer of wireless, while at price £549 the opposite was the case. Having found evidence of a significant interaction effect the additive model fitted in the previous example will not be adequate.

Table 8.6 gives the mean sales corresponding to each of the $3 \times 2 = 6$ FLCs. The Overall Mean Sales for the experiment was 97 laptops per group.

Table 8.7 gives the main effects for the offer factor. Note that the effects sum to zero.

Table 8.8 gives the main effects for the price factor. Note that the effects sum to zero. As before, we have

$$\text{Data} = \text{Fit} + \text{Residual}.$$

In this case we take

$$\text{Fit} = \text{Overall mean} + \text{Price effect} + \text{Offer effect} + \text{Interaction effect}.$$

The interaction effect is obtained by computing, for each FLC, the value of Overall mean + Price effect + Offer effect and choosing the interaction effect to be such that the fit
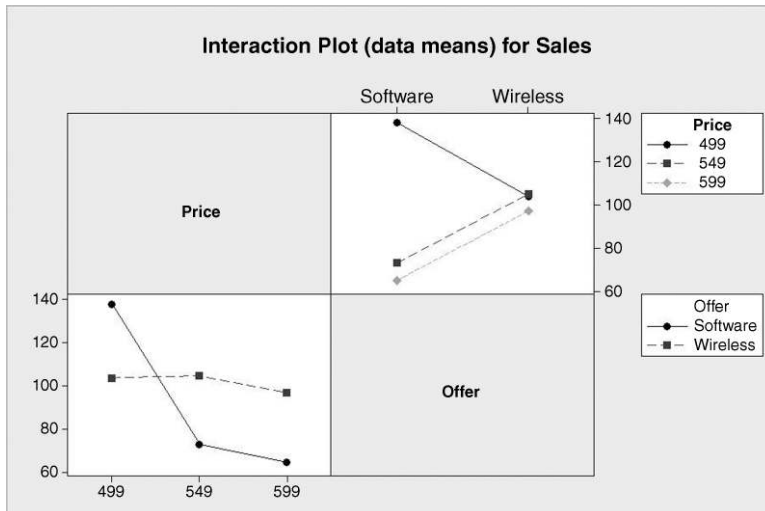

Figure 8.13   Interaction plots of marketing data.

Table 8.6   Mean sales for marketing experiment.

Price	Offer	
	Software	Wireless
£499	138	104
£549	73	105
£599	65	97

Table 8.7   Effects for offer.

Level	Mean sales	Effect
Software	92	− 5
Wireless	102	5

equals the mean sales in the experiment for the corresponding FLC. The detail is set out in Table 8.9. Note the mathematical symbols between the component sub-tables – the corresponding cell entries in the first four sub-tables are added together to give the corresponding cell entries in the final table.

For example, with level £499 for price and level software for offer, the sum of overall mean, price effect and offer effect equals $97 + 24 + (-5) = 116$. To obtain the mean sales figure of 138 for this FLC requires addition of a price–offer interaction effect of 22. With level £499 for price and level wireless for offer, the sum of overall mean, price effect and offer effect equals $97 + 24 + 5 = 126$. To obtain the mean sales figure of 104 for this combination of factor levels requires addition of a price–offer interaction effect of -22. The reader is invited to check the remaining entries in the fourth sub-table of Table 8.9. Note that *both* the rows *and* the columns of the matrix of interaction effects sum to 0.

Residuals are calculated as usual using Residual $=$ Data $-$ Fit. With price £499 and offer of software the two observed sales figures were 136 and 140, so the corresponding residuals are $136 - 138 = -2$ and $140 - 138 = 2$. (Whenever there are just two replications the residuals for each FLC will be equal in magnitude and opposite in sign.)

The *fixed* effects model for a general two-factor design *with interaction* may be specified as shown in Box 8.2. The number of levels of the row factor is a, the number of levels of the column factor is b and the number of replications is n. For example, with $i = 1, j = 2$ and $k = 3$ we have the specific relationship

$$Y_{123} = \mu + \alpha_1 + \beta_2 + (\alpha\beta)_{12} + \varepsilon_{123}.$$

Table 8.8   Effects for price.

Level	Mean sales	Effect
£499	121	24
£549	89	− 8
£599	81	− 16

Table 8.9   Fitting the model.

Overall mean	Offer	
Price	Software	Wireless
£499	97	97
£549	97	97
£599	97	97

+

Price effect	Offer	
Price	Software	Wireless
£499	24	24
£549	− 8	− 8
£599	− 16	− 16

+

Offer effect	Offer	
Price	Software	Wireless
£499	− 5	5
£549	− 5	5
£599	− 5	5

+

Interaction effect	Offer	
Price	Software	Wireless
£499	22	− 22
£549	− 11	11
£599	− 11	11

=

Cell mean/fitted value	Offer	
Price	Software	Wireless
£499	138	104
£549	73	105
£599	65	97

Observed data value = Overall mean + Row factor effect

 + Column factor effect + Interaction effect + Random error

$$Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk}, \ \ i = 1, 2, \ldots, a, \ \ j = 1, 2, \ldots, b, \ \ k = 1, 2, \ldots, n$$

$$\sum_{i=1}^{a} \alpha_i = 0, \ \ \sum_{i=1}^{b} \beta_i = 0, \ \ \sum_{j=1}^{b} (\alpha\beta)_{ij} = 0, \ \ \sum_{i=1}^{a} (\alpha\beta)_{ij} = 0, \ \ \varepsilon_{ijk} \sim N(0, \sigma^2)$$

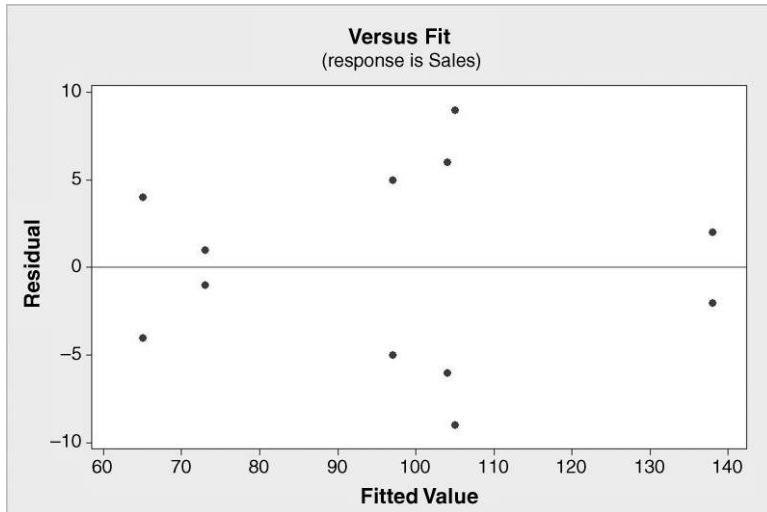**Box 8.2**   Fixed-effect model for two factors with interaction.

Figure 8.14   Plot of residuals against fits.

For the marketing experiment this states that the sales to the third group ($k = 3$, indicating the third replicate) with price £499 ($i = 1$, indicating the first level of the price factor) and with the offer of a wireless module ($j = 2$, indicating the second level of the offer factor) is made up of the overall mean plus the effect of price £499 plus the effect of the offer of wireless plus the interaction effect for level £499 of the price factor coupled with level wireless for the offer factor plus the random error component. Earlier we estimated the overall mean as 97, the effect of price £499 as 24, the effect of offer wireless as 5 and the interaction effect, $(\alpha\beta)_{12}$, for this combination of factor levels, as -22. Thus the fitted value for the combination of price £599 with offer wireless is the sum $97 + 24 + 5 + (-22) = 104$.

The value $s = 7.371$, which is given beneath the ANOVA table and is the square root of the error MS in the ANOVA table, provides an estimate of the standard deviation σ of the random error component of the model. The model predicts that the population of sales to groups obtained with the combination of price £499 with the wireless offer will be normally distributed with mean 104 and standard deviation 7.371. Similar statements may be made about the five other FLCs.

The response, sales, is a discrete random variable so, strictly speaking, cannot be normally distributed for a particular combination of factor levels. However, a normal probability plot of the residuals is satisfactory so one can be satisfied that the assumption of normality, underlying the valid use of the F-distribution to compute P-values, is approximately true. The plot of residuals against fits is shown in Figure 8.14.

The symmetry of the plot about the horizontal reference line, corresponding to residual value 0, stems from the fact noted earlier that, with two replications, the residuals occur in pairs of values with equal magnitude. The reader might feel that the wide variation in spread of the pairs of points might cast doubt on the model assumption of a random error with constant variance. However, use of **Stat > ANOVA > Test for Equal Variances. . .**with **Response: Sales** and **Factors:** Price Offer, yields a P-value of 0.668. Thus there is no evidence from Bartlett's test to cast doubt on a random error with constant variance. The main effects plot is shown in Figure 8.15.
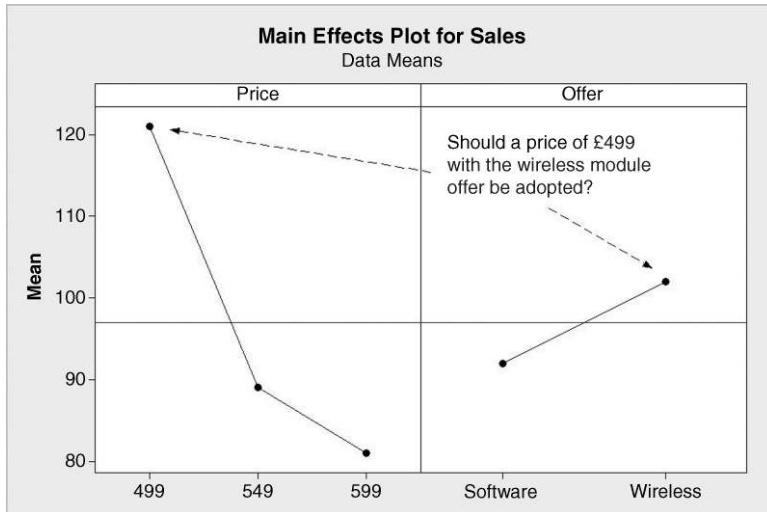
Figure 8.15 Main effects plot for marketing experiment.

In the case of the popcorn experiment, the best combination of factor levels, in terms of maximizing yield, could be identified by selecting the levels air and luxury on scrutiny of the main effects plot in Figure 8.5 for the levels of the factors corresponding to the greatest mean yields in each panel. In the case of the marketing experiment, scrutiny of the plot in Figure 8.15 might suggest the combination of price £499 with offer of wireless would be best, but, because of the interaction, it would appear that the combination of price £499 and offer of software is actually best – see Figure 8.12 and Table 8.6.

Grouping information from follow-up comparisons via the Tukey method with overall confidence level of 95% was obtained via **Stat > ANOVA > General Linear Model…**. The dialog is shown in Figure 8.16. In **Model:** we are communicating to Minitab the nature of the model we are using, i.e. $Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk}$. By entering Price, Offer and Price*Offer in **Model:** we are indicating the $\alpha_i + \beta_j + (\alpha\beta)_{ij}$ terms at the core of the model. In the **Comparisons** subdialog **Pairwise comparisons** were selected, by the **Tukey** method. **Grouping information**, with **Confidence level:** 95.0 was checked. In the **Terms:** window Price*Offer was entered as interest centres on sales from the different FLCs, i.e. price–offer combinations. Part of the corresponding section of the Session window output is shown Panel 8.5.

Note that the FLC of price £499 and software offer 'tops the league' with the highest observed mean sales of 138. This FLC comprises grouping A, and as this letter is not shared with any other grouping there is formal evidence of the superiority of the combination of price £499 and software offer over all other FLCs, in terms of the response, sales. Use of the grouping information option means that the Minitab user does not need to scrutinize and interpret a whole series of confidence intervals

8.1.4 The random effects model, components of variance

The *random* effects model for a general two-factor design *with interaction* may be specified as detailed in Box 8.3. The number of levels of the row factor is *a*, the number of levels of the
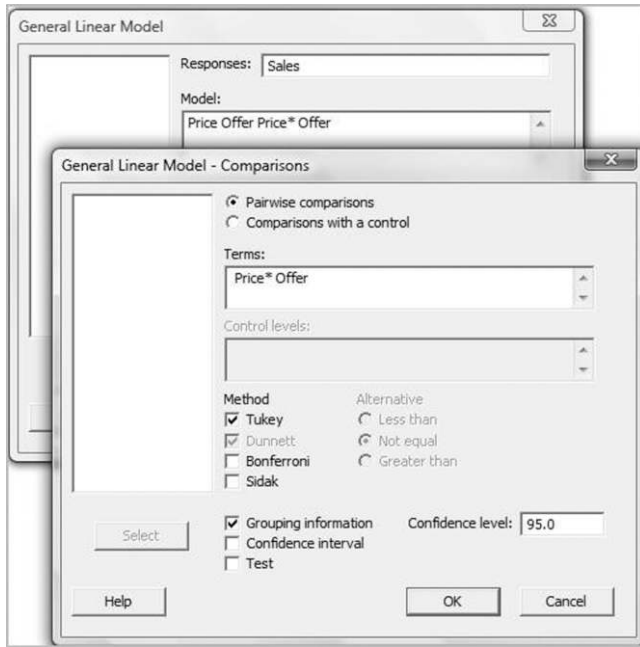
Figure 8.16   General Linear Model dialog.

column factor is b and the number of replications is n. In this model the effects are random variables – hence the terminology 'random effects model'. There are three null hypotheses that may be tested against alternatives:

$$H_0 : \sigma_a^2 = 0, \quad H_1 : \sigma_a^2 \neq 0;$$
$$H_0 : \sigma_b^2 = 0, \quad H_1 : \sigma_b^2 \neq 0;$$
$$H_0 : \sigma_{ab}^2 = 0, \quad H_1 : \sigma_{ab}^2 \neq 0.$$

In words, these null hypotheses state that there is no component of variation due to the row factor, the column factor, and the interaction, respectively.

```
Grouping Information Using Tukey Method and 95.0% Confidence

Price   Offer     N   Mean   Grouping
499     Software  2   138.0  A
549     Wireless  2   105.0     B
499     Wireless  2   104.0     B
599     Wireless  2    97.0     B C
549     Software  2    73.0       C D
599     Software  2    65.0         D

Means that do not share a letter are significantly different.
```

Panel 8.5   Session window output for follow-up comparisons.

Observed data value = Overall mean + Row factor effect + Column factor effect

+ Interaction effect + Random error

$Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk}, \ i = 1, 2, \ldots, a, \ j = 1, 2, \ldots, b, \ k = 1, 2, \ldots, n$

$\alpha_i \sim N(0, \sigma_a^2), \ \beta_j \sim N(0, \sigma_b^2), \ (\alpha\beta)_{ij} \sim N(0, \sigma_{ab}^2), \ \varepsilon_{ijk} \sim N(0, \sigma^2)$

(The random variables α_i, β_j, $(\alpha\beta)_{ij}$ and ε_{ijk} are independent.)

Box 8.3   Random effects model.

As an example, consider an experiment in which three operators, selected at random from a pool of operators, each measured the height (mm) of each one of a random sample of ten bottles of a particular type on two occasions. On each occasion the bottles were presented to the operators in a random sequence and on the second occasion the operators were unaware of the results they had obtained on the first occasion. All measurements were taken with the same calibrated height gauge under similar environmental conditions. The data are available in Heights.xls and a segment of the data is shown in Figure 8.17. (Data reproduced by permission of Ardagh Glass Ltd., Barnsley.)

In this scenario the response, Y, is height and the factors are bottle (X_1) and operator (X_2). Both factors are random. Analysis via Minitab cannot be carried out using **Stat > ANOVA > Two-Way...** because it only deals with the case of *fixed* effects. However, the analysis may be carried out using **Stat > ANOVA > Balanced ANOVA...**. Under **Results...** the option to **Display expected mean squares and variance components** was checked. Under **Graphs...** the options to create a normal probability plot of residuals and a plot of residuals versus fitted values were accepted. The remainder of the dialog, in which the model and the information that the bottle and operator factors are random are specified, is shown in Figure 8.18.

The ANOVA table, preceded by a list specifying the factors, types and levels from the Session window output, is displayed in Panel 8.6. The following conclusions may be reached concerning the hypotheses specified earlier:

- $H_0 : \sigma_a^2 = 0$ is rejected in favour of $H_1 : \sigma_a^2 \neq 0$ at the 0.1% level of significance (P-value given as 0.000 to three decimal places);

- $H_0 : \sigma_b^2 = 0$ is rejected in favour of $H_1 : \sigma_b^2 \neq 0$ at the 1% level of significance (P-value = 0.002);

- $H_0 : \sigma_{ab}^2 = 0$ cannot be rejected (P-value = 0.621).

The normal plot of the residuals and the plot of residuals versus fits were considered to be satisfactory and are not reproduced in this text.

As in the case of the random effects model for a single factor, expressions can be derived, in terms of the four variances in the model, for the expected mean squares. Use of these expressions gives rise to estimates of each of the four variances. The expressions and the estimates from the Session window output are shown in Panel 8.7. The estimate of σ_{ab}^2 is -0.00002. A negative variance is impossible. Together with the P-value of 0.621 for interaction (Panel 8.6), the negative estimate provides a further indication that the interaction component should be dropped from the model. Thus the model was revised by removing the

↓	C1	C2-T	C3
	Bottle	Operator	Height
1	1	Neil	214.82
2	2	Neil	214.61
3	3	Neil	214.52
4	4	Neil	214.57
5	5	Neil	214.64
6	6	Neil	214.72
7	7	Neil	214.60
8	8	Neil	214.73
9	9	Neil	214.70
10	10	Neil	214.80
11	1	Neil	214.83
12	2	Neil	214.64
13	3	Neil	214.53
14	4	Neil	214.61
15	5	Neil	214.64
16	6	Neil	214.73
17	7	Neil	214.61
18	8	Neil	214.75
19	9	Neil	214.67
20	10	Neil	214.78

Figure 8.17   Segment of bottle height data.

bottle–operator interaction term from the model by specifying **Model:** Bottle Operator in the dialog, i.e. by deleting Bottle*Operator from the window. Revised Session window output is shown in Panel 8.8.

The final model is shown in Box 8.4. The variance components σ_a^2, σ_b^2 and σ^2, due to bottle, operator and random error respectively, are estimated as 0.009 34, 0.000 09 and 0.000 23 respectively. Since any observed data value involves a sum of independent random variables the result in Box 4.2 in Section 4.3.1 is applicable. The variance of Y_{ijk} is thus $0 + \sigma_a^2 + \sigma_b^2 + \sigma^2$ (μ, being a constant, has variance 0). Thus the estimated variance of Y_{ijk} is given by

$$0.009\,34 + 0.000\,09 + 0.000\,23 = 0.009\,66$$

and the estimated proportion of total variance accounted for by bottle is 0.009 34/ 0.009 66 = 96.7%. The fact that a relatively large proportion of the total variability is attributable to the product is desirable from the point of view of the performance of the measurement system. The topic of measurement system analysis will be considered in detail in Chapter 9.
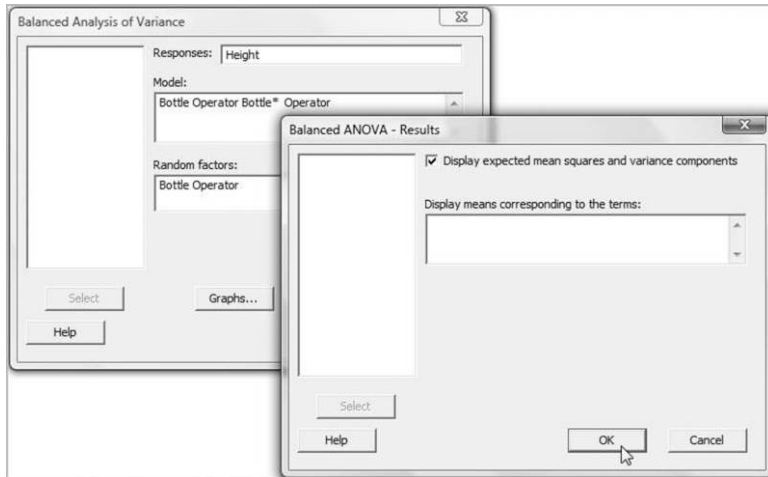
Figure 8.18 Dialog for balanced ANOVA.

```
ANOVA: Height versus Bottle, Operator

Factor      Type     Levels  Values
Bottle      random       10  1,  2,  3,  4,  5,  6,  7,  8,  9, 10
Operator    random        3  Lee, Neil, Paul


Analysis of Variance for Height

Source           DF        SS        MS       F      P
Bottle            9  0.506302  0.056256  273.43  0.000
Operator          2  0.003863  0.001932    9.39  0.002
Bottle*Operator  18  0.003703  0.000206    0.86  0.621
Error            30  0.007150  0.000238
Total            59  0.521018


S = 0.0154380   R-Sq = 98.63%   R-Sq(adj) = 97.30%
```

Panel 8.6 ANOVA table.

```
                                 Expected Mean Square
                   Variance  Error  for Each Term (using
   Source          component  term  unrestricted model)
1  Bottle            0.00934     3   (4) + 2 (3) + 6 (1)
2  Operator          0.00009     3   (4) + 2 (3) + 20 (2)
3  Bottle*Operator  -0.00002     4   (4) + 2 (3)
4  Error             0.00024         (4)
```

Panel 8.7 Estimated variance components.

	Source	Variance component	Error term	Expected Mean Square for Each Term (using unrestricted model)
1	Bottle	0.00934	3	(3) + 6 (1)
2	Operator	0.00009	3	(3) + 20 (2)
3	Error	0.00023		(3)

Panel 8.8   Estimated variance components from revised model.

8.2   Full factorial experiments in the 2^k series

8.2.1 2^2 Factorial experimental designs, display and analysis of data

Factorial experiments in which all the factors of interest have just two levels are of particular importance in the improve phase of many Six Sigma projects. An experiment involving, for example, three factors, each with two levels, involves a total of $2 \times 2 \times 2$ or 2^3 FLCs. A 2^k factorial experiment involves k factors, each with two levels.

Consider the manufacture of a product, for use in the making of paint, in a batch process. Fixed amounts of raw material are heated under pressure in reactor 1 for a fixed period of time and the product is then recovered. Currently the process is operated at temperature 225 °C and pressure 4.5 bar. As part of a Six Sigma project, aimed at increasing product yield, a 2^2 factorial experiment with two replications was planned. Yields from the process with current temperature and pressure levels average 90 kg. It was decided after discussion amongst the project team to use the levels 200 °C and 250 °C for temperature and the levels 4.0 bar and 5.0 bar for pressure.

Use was then made of **Stat > DOE > Factorial > Create Factorial Design. . .** with the option **2-level factorial (default generators)** selected under **Type of Design**. Part of the dialog is shown in Figure 8.19. **Number of factors:** was specified as 2. Under **Designs . . .** the only available design is the full factorial design. The number of FLCs is $2^2 = 4$, so Minitab indicates this by listing the number of **Runs** required as 4. **Resolution**, Full in this case, will be discussed later in the chapter. Minitab represents 2^2 as 2**2. The **Number of replicates:** was specified as 2, the **Number of blocks:** as 1 and **Number of center points:** as 0. (Experimental designs involving centre points will be considered in Chapter 10.)

Observed data value = Overall mean + Bottle effect + Operator effect

+ Random error

$$Y_{ijk} = \mu + \alpha_i + \beta_j + \varepsilon_{ijk}, \ i = 1, 2, \ldots, a, \ j = 1, 2, \ldots, b, \ k = 1, 2, \ldots, n$$

$$\alpha_i \sim N(0, \sigma_a^2), \ \beta_j \sim N(0, \sigma_b^2), \quad \varepsilon_{ijk} \sim N(0, \sigma^2)$$

(The random variables α_i, β_j and ε_{ijk} are independent.)

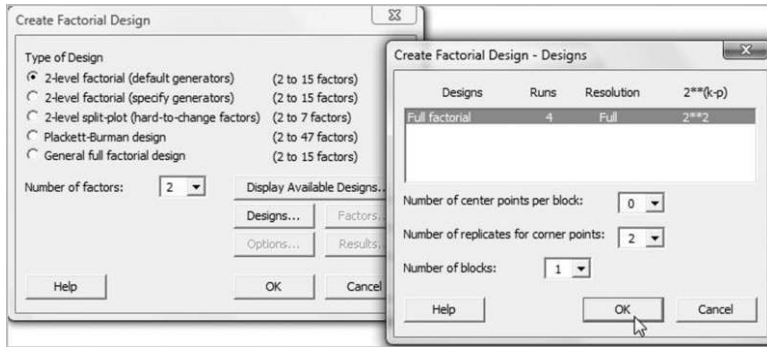**Box 8.4**   Revised random effects model.

Figure 8.19   Dialog for design of a 2^2 experiment.

On clicking **OK** one may commence the remainder of the dialog. Under **Factors. . .** the factors temperature and pressure (both numeric in this case) and their levels were specified. The defaults were accepted under **Options. . .**, i.e. to **Randomize runs** and to **Store design in worksheet**. (No reference will be made in this book to **Fold Design** and **Fraction**.) Under **Results. . .** the defaults were also accepted. On clicking **OK**, **OK** the resulting worksheet should be augmented with a column in which the values of yield obtained can be recorded and a column in which those carrying out the experiment can note any unusual happening or information that might prove relevant to the analysis of the data from the experiment and to the project. Before commencing the experimentation the work to date should be stored in a Minitab project. Fictitious yield data are used in order to make the checking of key calculations straightforward for the reader.

Once the experiment has been completed one could carry out the display and analysis using facilities for plotting and analysis available under **Stat > ANOVA > . . .**, but Minitab has built-in facilities for the analysis of 2^k factorial experiments via **Stat > DOE > . . .**. For initial display of the data one may use **Stat > DOE > Factorial > Factorial Plots. . .**. The data and dialog involved in the creation of the three plots available are shown in Figure 8.20. The data are provided in Reactor1.MTW and the reader should use that worksheet should he/she wish to re-create the displays and analysis that follows as the worksheet, which was created using **Stat > DOE > Factorial > Create Factorial Design. . .**, includes hidden stored information on the experimental design.

Main Effects Plot, **Interaction Plot** and **Cube Plot** were all checked. Once a plot type has been checked one must then click on the corresponding **Setup. . .** button and enter the **Responses** to be plotted and **Factors to Include in Plots**. The arrow keys may be used to select or deselect highlighted factors from the list of **Available:** factors. Both temperature and pressure were selected for all three plots, the defaults. For all three plots the default option to display **Data Means** as **Type of Means to Use in Plots** was accepted. In the case of the interaction plot the option to **Draw full interaction plot matrix** was selected in order to obtain the display in Figure 8.22. (The reader should note that specification of a response is optional in the case of the cube plot. This allows experimenters, who wish to do so, to create a blank cube plot on which they can record data means for discussion prior to any formal analysis using software.)

The left-hand component of the annotated main effects plot in Figure 8.21 has the mouse pointer located at the point representing the mean yield of 88 kg from all the experimental runs
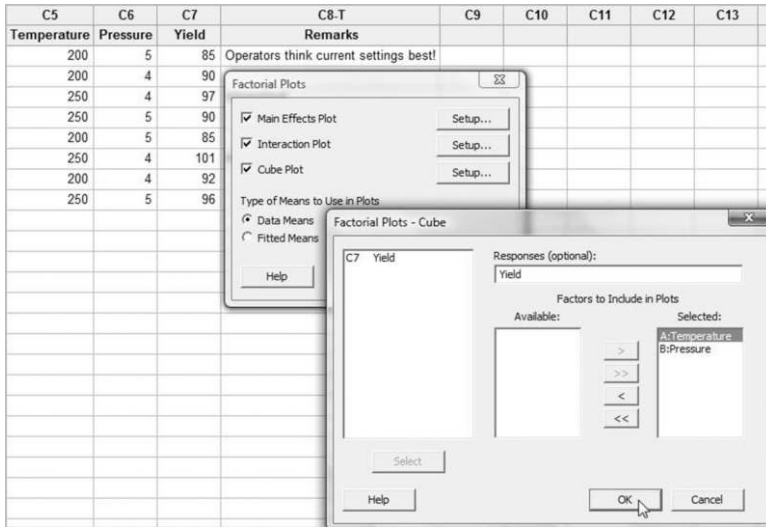
Figure 8.20   Dialog for creating factorial plots.

carried out with temperature 200 °C, as can readily be verified from the data in the worksheet in Figure 8.20. The mean yield from all the experimental runs carried out with temperature 250 °C was 96 kg. Hence, on average, increasing temperature from 200 °C to 250 °C increases yield of product by 8 kg. The reader is invited to confirm that the right-hand component of the plot indicates that, on average, increasing pressure from 4 bar to 5 bar decreases yield of product by 6 kg. The main effect of temperature is 8 kg and the main effect of pressure is − 6 kg.

Both versions of the interaction plot are shown in Figure 8.22. The mouse pointer is placed over the point in one version that corresponds to the FLC with temperature 250 °C and pressure
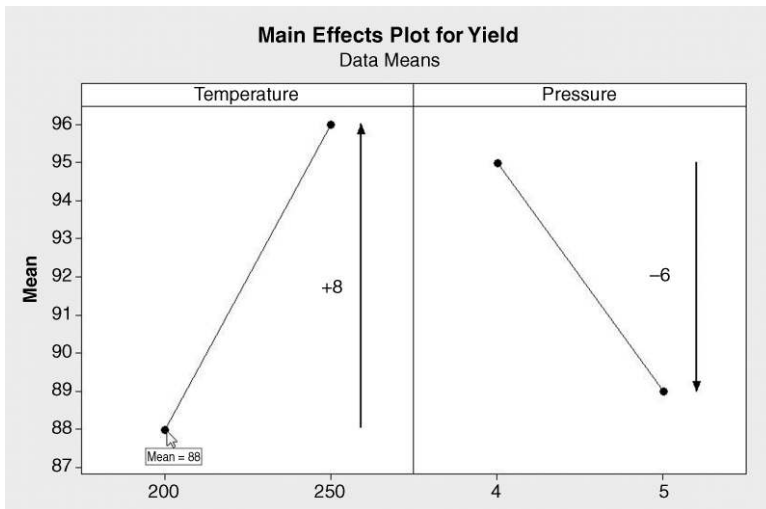
Figure 8.21   Main effects plot for reactor 1.
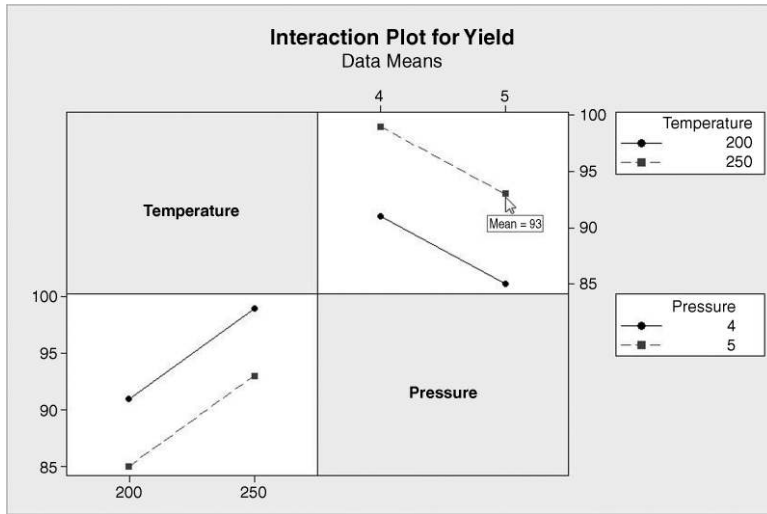
Figure 8.22   Interaction plots for reactor 1.

5 bar. Scrutiny of the data table visible in the worksheet in Figure 8.20 reveals yields of 90 and 96 kg with mean 93 kg. The parallel lines indicate no temperature–pressure interaction in this situation. The reader is encouraged to create the plot in Figure 8.22 for him/herself and, with a copy of the data at hand, to move the mouse pointer to each of the eight points in turn and confirm the mean yields displayed.

The mean yields for the four FLCs used in the experiment are plotted at the vertices of a square (a cube in two dimensions!) in the cube plot in Figure 8.23. The annotated version of the cube plot in Figure 8.24 indicates another way of determining the main effect of temperature. The annotated version of the cube plot in Figure 8.25 indicates another way of determining the main effect of pressure.
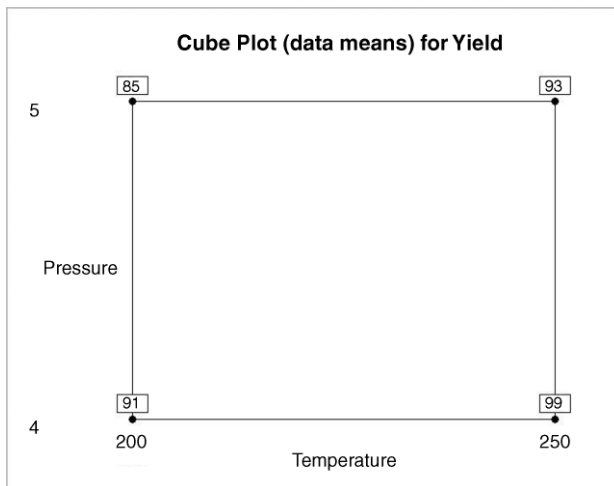

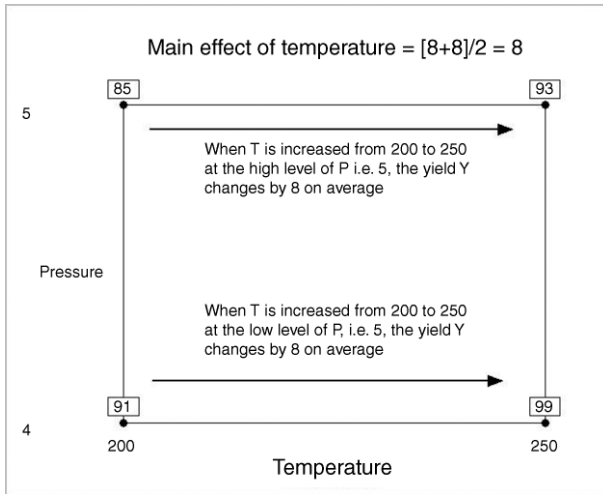**Figure 8.23**   Cube plot for reactor 1.

Figure 8.24   Annotated cube plot for main effect of temperature.

Having displayed the data in various ways, the next step is to perform the formal analysis using **Stat** > **DOE** > **Factorial** > **Analyze Factorial Design. . ..** (Note that the Minitab icon for **Analyze Factorial Design. . .** is based on the sort of cube plot we have been considering in Figure 8.23.) The dialog is shown in Figure 8.26.

It is necessary to specify Yield under **Response:** and to specify the model using **Terms. . ..** Here under **Terms:** A: Temperature, B: Pressure and AB, denoting the temperature–pressure interaction, are selected as the default. Part of the Session window output obtained is shown in Panel 8.9.
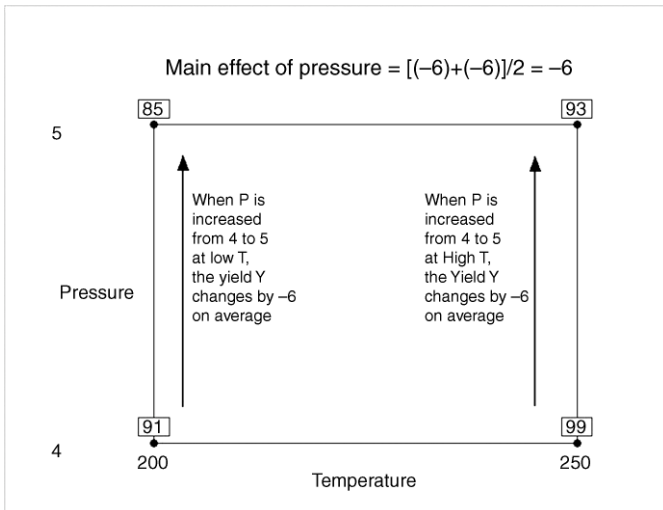

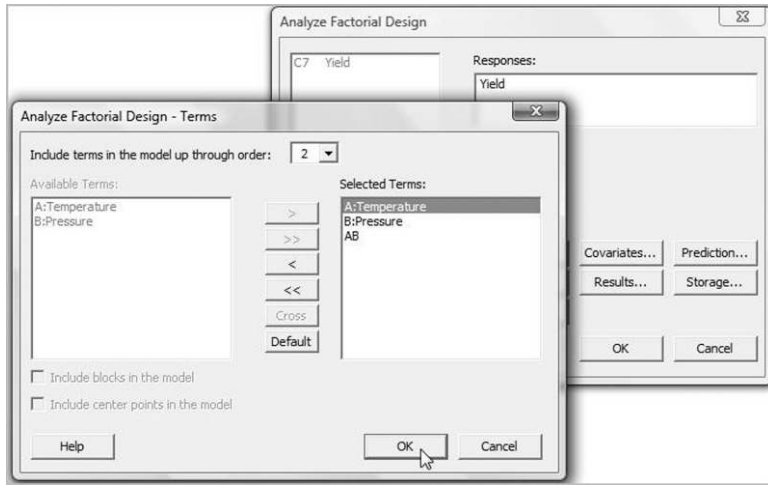**Figure 8.25**   Annotated cube plot for main effect of pressure.

Figure 8.26 Analysing the factorial experiment.

Note that the temperature effect of 8 and the pressure effect of –6 estimated from the data earlier are given in the first column. The interaction plots indicated no temperature–pressure interaction and this corresponds to the interaction effect estimate of zero. (The redundant negative sign will have arisen due to the way that numbers are stored in the software.) The Coef (coefficient) column in the output has the value 92 as the first entry. This is the overall mean yield for the entire experiment and is referred to as the constant term in the model. The coefficients corresponding to temperature, pressure and temperature $\times$ pressure are simply half the corresponding effects. (Equations in which these coefficients are required will be discussed later in the chapter.) The heading SE Coef refers to the standard errors of the coefficients. For example, the constant term 92 is the mean of a sample of eight measurements. The standard error or standard deviation of a mean of eight values is the standard deviation for individual values divided by $\sqrt{8}$. The output gives the estimated standard deviation of individual yields obtained from the ANOVA to be $s = 2.645\,75$. Division by $\sqrt{8}$ yields 0.9354. The values of the coefficients divided by their standard errors give a Student's t-statistic for testing the null hypothesis that each coefficient is zero. The P-values for each are given in the final column. The fact that the P-values for both temperature and pressure are less than 0.05 may be taken as evidence that both temperature and pressure have a real effect on

```
Estimated Effects and Coefficients for Yield (coded units)

Term                  Effect    Coef  SE Coef      T      P
Constant                      92.000   0.9354  98.35  0.000
Temperature            8.000    4.000   0.9354   4.28  0.013
Pressure              -6.000   -3.000   0.9354  -3.21  0.033
Temperature*Pressure  -0.000   -0.000   0.9354  -0.00  1.000


S = 2.64575     PRESS = 112
R-Sq = 87.72%   R-Sq(pred) = 50.88%   R-Sq(adj) = 78.51%
```

Panel 8.9 ANOVA for 2^2 factorial experiment on reactor 1.

yield. However, as suspected from viewing the interaction plots, there is no evidence of a nonzero temperature–pressure effect.

Consider now an experiment, with the same design as above, performed on reactor 2, a different type from reactor 1. Currently, as with reactor 1, the process is also operated at temperature 225 °C and pressure 4.5 bar, and yields average 90 kg. The data are available in Reator2.MTW, and the reader is invited to create a cube plot for this second experiment and to verify that the main effects for temperature and pressure are 8 and -6, respectively. One version of the interaction plot for reactor 2 is displayed, with annotation, in Figure 8.27. The nonparallelism suggests the presence of a temperature–pressure interaction effect.

- When pressure was set at 5 bar the effect of increasing temperature from 200 to 250 °C was to increase mean yield, on average, from 87 to 91 kg, i.e. by 4 kg.

- When pressure was set at 4 bar the effect of increasing temperature from 200 to 250 °C was to increase mean yield, on average, from 89 to 101 kg, i.e. by 12 kg.

- The main effect of temperature is given by $(4 + 12)/2 = 8$.

- The temperature–pressure interaction effect is given by $(4 - 12)/2 = -4$.

In the final calculation of the interaction effect it is important to note that the difference considered is that of the effect of temperature at the higher level of pressure less that of the effect at the lower level of pressure. (Had the changes in mean yield, indicated by the arrows in Figure 8.27, been equal then the line segments linking points with the same level of temperature in the interaction plot would have been parallel, as in the top right-hand panel in Figure 8.22. The difference between the changes would be zero and the interaction effect would be zero.) The second version of the interaction plot is shown in Figure 8.28.
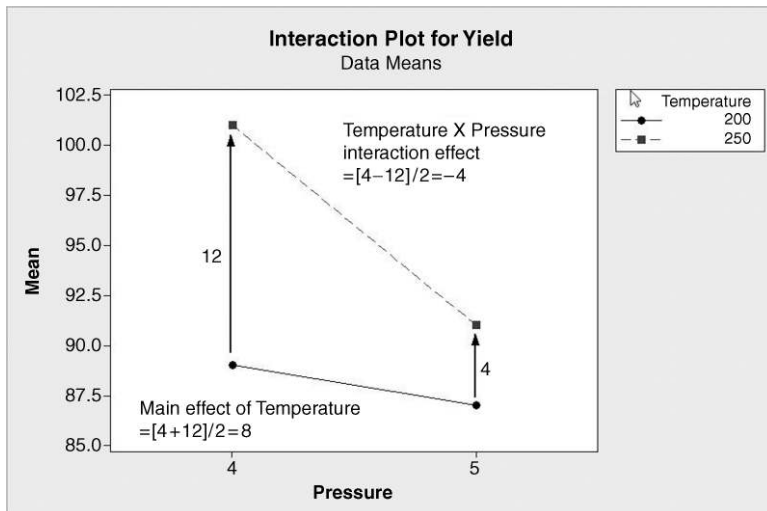

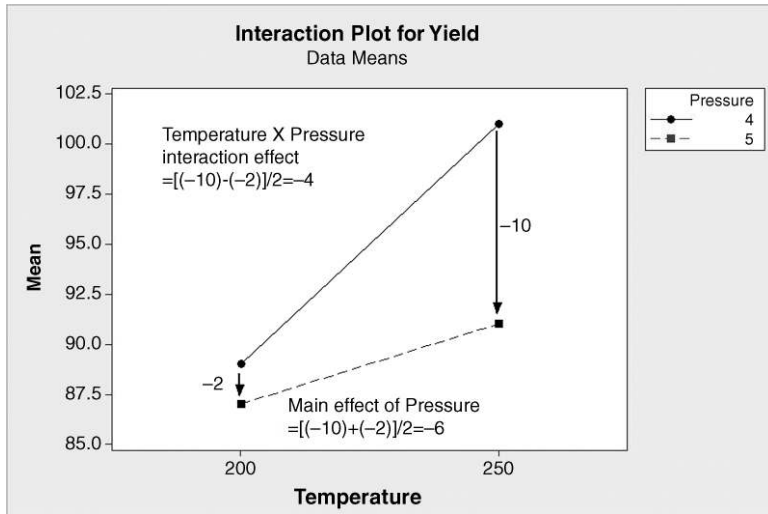**Figure 8.27**   First version of interaction plot for reactor 2.

Figure 8.28   Second version of interaction plot for reactor 2.

- When temperature was set at 250 °C the effect of increasing pressure from 4 to 5 bar was to change mean yield, on average, from 101 to 91 kg, i.e. by -10 kg.

- When temperature was set at 200 °C the effect of increasing pressure from 4 to 5 bar was to change mean yield, on average, from 89 to 87 kg, i.e. by -2 kg.

- The main effect of pressure is given by $[(-10) + (-2)]/2 = -6$.

- The temperature-pressure interaction effect is given by $[(-10) - (-2)]/2 = -4$.

The reader is also invited to verify using **Stat > DOE > Factorial > Analyze Factorial Design...** that the output in Panel 8.10 is obtained. Note that the effects calculated earlier are confirmed and also that the interaction effect differs significantly from zero at the 10% level of significance. Thus there is slight evidence of a real temperature–pressure interaction in the case of reactor 2.

Montgomery (2009, p. 566) gives an example from the electronics industry. It concerned registration notches cut in printed circuit boards using a router. Although the process was stable, producing a satisfactory mean notch size, variability was too high. High variability leads to problems in assembly, when components are being placed on the boards, due to

Factorial Fit: Yield versus Temperature, Pressure

```
Estimated Effects and Coefficients for Yield (coded units)

Term                  Effect    Coef  SE Coef       T      P
Constant                      92.000   0.7906  116.37  0.000
Temperature            8.000   4.000   0.7906    5.06  0.007
Pressure              -6.000  -3.000   0.7906   -3.79  0.019
Temperature*Pressure  -4.000  -2.000   0.7906   -2.53  0.065
```

Panel 8.10   ANOVA for 2^2 factorial experiment on reactor 2.

improper registration. The engineers associated with the process reckoned that the high variability was due to vibration and decided to use a 2^2 factorial experiment, with *four* replications, to investigate the effect of the factors bit size and speed on the response vibration, as measured using accelerometers attached to the boards. The reason for choosing vibration as the response was that variability in notch dimension was difficult to measure directly but it was known that vibration correlated positively with variability. The levels were 1/16 inch and 1/8 inch for bit size, and 40 rpm and 80 rpm for speed. Vibration was measured in cycles per second. The design and the data, reproduced by permission of John Wiley & Sons, Inc., New York, are stored in Vibration.MTW.

Use of **Stat** > **DOE** > **Factorial** > **Analyze Factorial Design. . .** yields the output in Panel 8.11. All *P*-values are less than 0.001, which indicates, in particular, that we have strong evidence of interaction between bit size and speed.

The main effects plot is shown in Figure 8.29. Naïve scrutiny of the main effects plot would suggest that the combination of small bit size with low speed would be best in terms of keeping vibration as low as possible. However, running the process using low speed would have had major implications in terms of process throughput. The major interaction between bit size and speed provided a resolution to this issue. The interaction plots are presented in Figure 8.30. These plots reveal that, with bit size 1/16 inch, vibration is low at both speeds 40 rpm and 80 rpm. Thus it was decided to operate the process with bit size 1/16 inch and speed 80 rpm. This led to a major reduction in the variability of the size of the registration notches.

The residual plots are shown in Figure 8.31. They were created using **Graphs. . .** in **Analyze Factorial Design. . .** with the **Four in one** option selected. Although the variability of the set of residuals for the lowest of the four fitted values appears relatively low, use of **Stat** > **ANOVA** > **Test for Equal Variances. . .** with **Response:** Vibration and **Factors:** Bit Size Speed yields no formal evidence of nonconstant variability. The plots appear to be generally satisfactory.

8.2.2 Models and associated displays

In order to introduce further important ideas some notation is required. Were one to design a 2^2 experiment with no centre points, one replication, no blocking, no randomization of the run order and using the factor names X1 and X2, with the default levels, then the worksheet displayed in Figure 8.32 would be obtained. The columns named X1 and X2 in the worksheet contain the essence of the design. The X1 × X2 interaction effect may be calculated by using an additional column X1X2. This additional X1X2 column may be obtained from the X1 and X2 columns by simply multiplying corresponding row entries together, hence the X1 × X2 notation.

Factorial Fit: Vibration versus Bit Size, Speed

Estimated Effects and Coefficients for Vibration (coded units)

Term	Effect	Coef	SE Coef	T	P
Constant		23.831	0.6112	38.99	0.000
Bit Size	16.638	8.319	0.6112	13.61	0.000
Speed	7.538	3.769	0.6112	6.17	0.000
Bit Size*Speed	8.713	4.356	0.6112	7.13	0.000

Panel 8.11   ANOVA for 2^2 factorial router experiment.
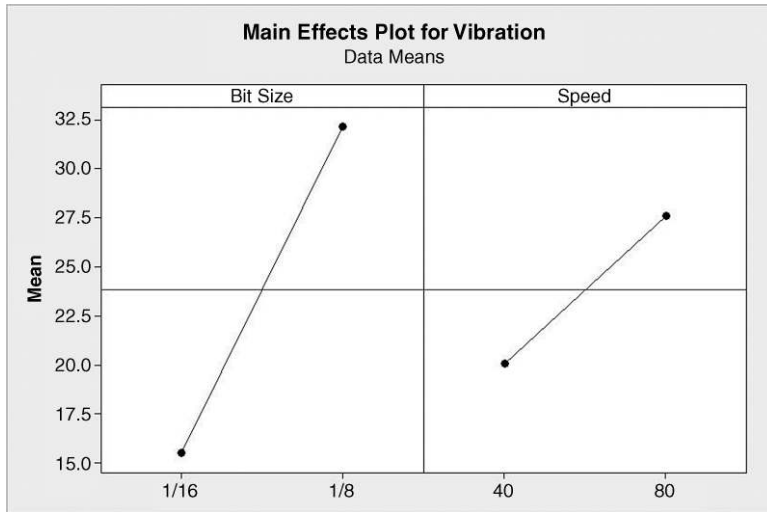
Figure 8.29   Main effects plot for router experiment.

In terms of the reactors referred to earlier, X1 and X2 may be considered as coded variables representing temperature and pressure, respectively. Recall that current operating conditions were temperature 225 °C and pressure 4.5 bar. These would have coded values X1 = 0 and X2 = 0, respectively. The high level of temperature used in the experiment, i.e. 250 °C, would be coded as X1 = 1 and the low level, 200 °C, as − 1. Similarly, the low and high levels of pressure used, 4 bar and 5 bar, would be coded as X2 = − 1 and X2 = 1, respectively. The author finds it helpful to think in terms of a reactor control panel as shown in Figure 8.33. The low temperature of 200 °C is 'one notch down' (X1 = − 1) from the current
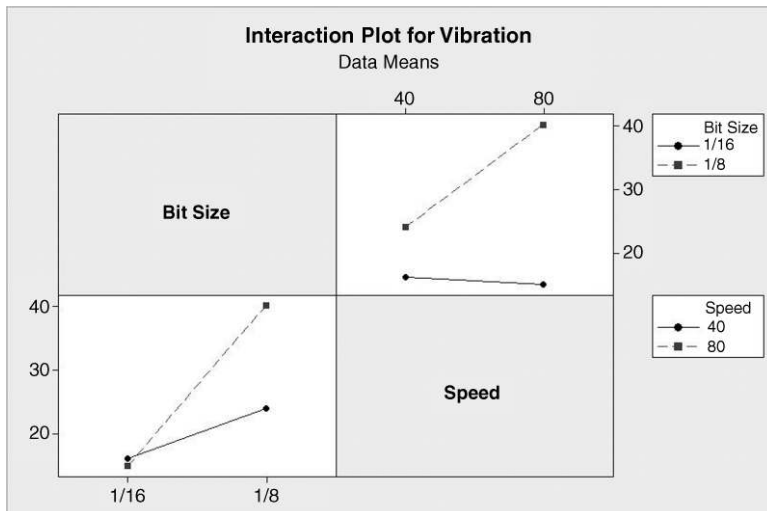

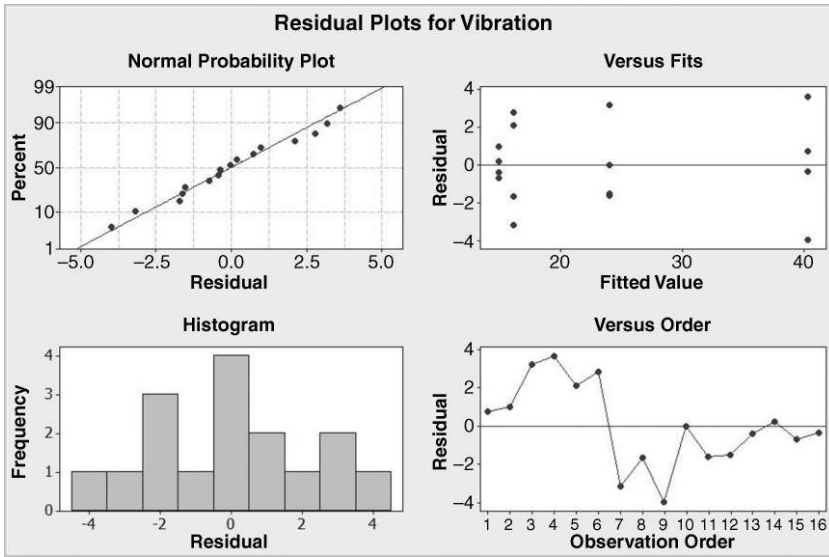**Figure 8.30**   Interaction plots for router experiment.

Figure 8.31   Residual plots for router experiment.

StdOrder	RunOrder	CenterPt	Blocks	X1	X2
3	1	1	1	-1	1
1	2	1	1	-1	-1
2	3	1	1	1	-1
4	4	1	1	1	1

Figure 8.32   Basic 2^2 design in standard order.
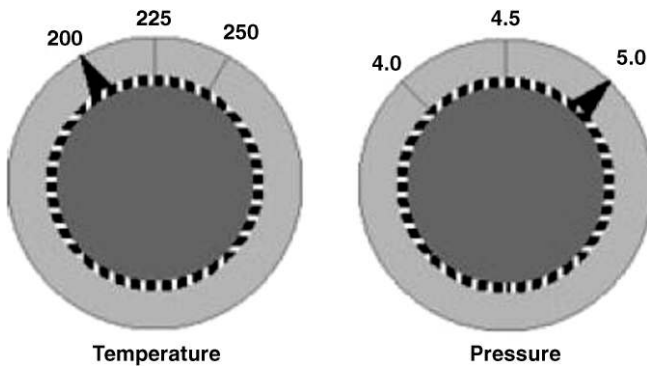
Figure 8.33   Reactor control panel.

Table 8.10   Reactor 1 data.

FLC	X1	X2	X1X2	Mean Y
1	-1	-1	1	91
2	1	-1	-1	99
3	-1	1	-1	85
4	1	1	1	93

operational setting and the high pressure of 5 bar is 'one notch up' (X2 = 1) from the current operational setting.

Table 8.10 lists the four FLCs used in the experiment with reactor 1. The means in the final column are the means of the pairs of yields obtained with each FLC. The overall mean and effects may be calculated as follows. The overall mean is simply $(91 + 99 + 85 + 93)/4 = 92$. The main effect of temperature is given by

$$[\text{Mean yield with X1} = 1] - [\text{Mean yield with X1} = -1]$$

$$= \frac{99 + 93}{2} - \frac{91 + 85}{2} = 96 - 88 = 8.$$

The main effect of pressure is

$$[\text{Mean yield with X1} = 1] - [\text{Mean yield with X2} = -1]$$

$$= \frac{85 + 93}{2} - \frac{91 + 99}{2} = 89 - 95 = -6.$$

Finally the temperature–pressure interaction effect is given by

$$[\text{Mean yield with X1X2} = 1] - [\text{Mean yield with X1X2} = -1]$$

$$= \frac{91 + 93}{2} - \frac{99 + 85}{2} = 92 - 92 = 0.$$

These calculations are summarized in Table 8.11. Each coefficient is half the corresponding effect.

Table 8.12 lists the four FLCs used in the experiment with reactor 2. The means in the final column are the means of the pairs of yields obtained with each FLC. The reader is invited to check the calculation of the three effects and corresponding coefficients displayed in Table 8.13 and to confirm that the overall mean is 92.

Table 8.11   Reactor 1 effects.

	Mean response at level		Effect	Coefficient
	$-$	$+$		
X1	88	96	8	4
X2	95	89	–6	–3
X1X2	92	92	0	0

Table 8.12   Reactor 2 data.

Reactor 2	Coded Factor Levels		X1X2	Mean Y
FLC	X1	X2		
1	-1	-1	1	89
2	1	-1	-1	101
3	-1	1	-1	87
4	1	1	1	91

The constant (overall mean) and the coefficients generated by Minitab provide statistical models for the expected response of the form

$$Y = \text{Overall mean} + \text{Coeff. of X1} \times \text{X1} + \text{Coeff. of X2} \times \text{X2}$$
$$+ \text{Coefficient of X1X2} \times \text{X1X2}.$$

For reactor1 we obtain

$$Y = 92 + 4X1 - 3X2 + 0X1X2 = 92 + 4X1 - 3X2,$$

and for reactor 2 we have

$$Y = 92 + 4X1 - 3X2 - 2X1X2$$

The models are of the form $Y = f(\mathbf{X})$ referred to in Section 1.4. Here the bold X represents the pair of factors X1 and X2. The model may be used to predict the expected yield for any FLC. (The models may also be obtained using multiple regression methods that will be referred to in Chapter 10.) As an example, consider reactor 1 operating at temperature 250 °C (X1 = 1) and pressure 4 bar (X2 = –1). Substitution of these values into the reactor 1 model equation gives expected yield

$$Y = 92 + 4 \times 1 - 3 \times (-1) = 92 + 4 + 3 = 99.$$

Of course this is the mean yield obtained for reactor 1 with temperature 250 °C and pressure 4 bar in the experiment and is the fitted value from the model for the particular FLC of temperature 250 °C and pressure 4 bar.

As a second example, consider reactor 2 operating at temperature 200 °C (X1 = –1) and pressure 5 bar (X2 = 1). Substitution of these values into the reactor 2 model equation gives expected yield

$$Y = 92 + 4 \times (-1) - 3 \times 1 - 2 \times (-1) \times 1 = 92 - 4 - 3 + 2 = 87.$$

Table 8.13   Reactor 2 effects.

	Mean response at level		Effect	Coefficient
	$-$	$+$		
X1	88	96	8	4
X2	95	89	-6	-3
X1X2	94	90	-4	-2

This is the mean yield obtained for reactor 2 with temperature 200 °C and pressure 5 bar in the experiment and is the fitted value from the model for that particular FLC.

In terms of optimizing the performance of the reactors by achieving as great a yield as possible, it would appear that both operate best, in the ranges of temperature and pressure considered in the experiments, at temperature 250 °C and pressure 4 bar. The ranges of factor levels considered in an experiment constitute the design space. With the reactor models above it is a fairly simple matter to substitute coded values of temperature and pressure into the model equations to obtain predicted responses.

Minitab provides a facility for investigation of process optimization using the models fitted to data from 2^k factorial experiments. With worksheet Reactor2.MTW open, for example, this facility is available via **Stat** > **DOE** > **Factorial** > **Response Optimizer. . ..** Recall that the reactors have been achieving yields averaging 90 kg with the current operating settings of 225 °C and 4.5 bar. Suppose that the project team had decided that it was desirable to achieve yields as high as possible, with target 105 kg on average and no worse than 95 kg on average. The Response Optimizer may be used to explore model predictions in a visual way and without having to perform calculations directly from the model equations. Part of the necessary dialog is shown in Figure 8.34.

It is necessary here to have **Selected:** Yield as the (only) response and under **Setup...** to set **Goal** to Maximize, **Lower** to 95 and **Target** to 105. No value is entered for **Upper** when seeking to maximize a response. **Weight** and **Importance** need not concern us when dealing with a single response. Under **Options. . .** check only **Optimization Plot**; there is no need to enter any **Starting value** for the factors.

The outcome from implementation is an interactive screen, part of which has been captured in Figure 8.35. The reader is advised to have this screen on display on his/her computer for the
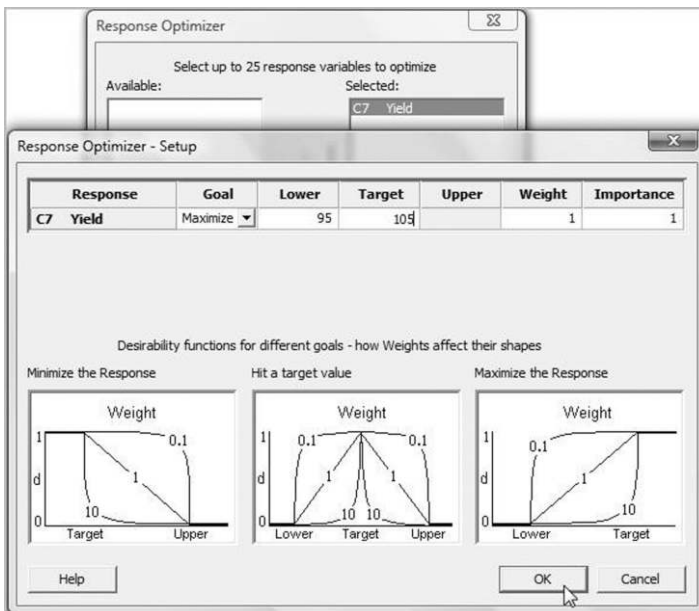

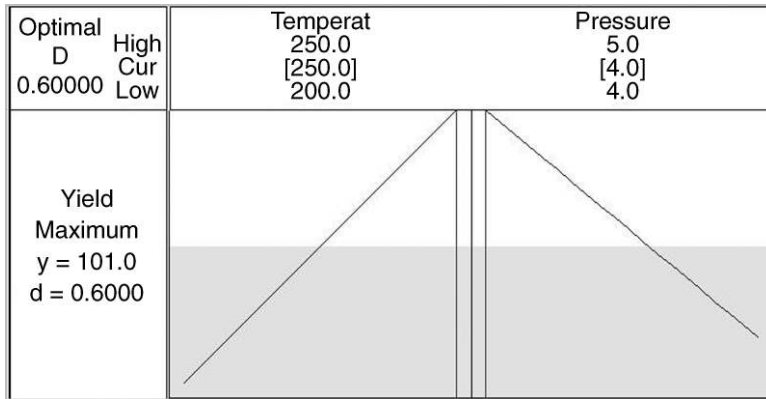**Figure 8.34**   Dialog for Response Optimizer.

Figure 8.35   Response Optimizer.

discussion that follows. Under the headings Temperat(ure) and Pressure the High and Low values specify the design space for the experiment. The Cur values in square brackets of 250.0 and 4.0 indicate that the maximum mean yield that can be achieved with reactor 2, based on the model fitted to the data from the experiment, will be obtained with temperature 250 °C and pressure 4 bar. The predicted maximum yield with this combination of factor levels is stated as $y = 101$. (Had the desired target of 105 been achieved then the desirability score, d, would have been 1.0. Since 101 lies 0.6 of the way between the lower value of 95 and the target value of 105 the desirability score is $d = 0.6$.)

One can explore predictions by changing the Cur(rent) temperature and pressure levels in one of two ways. The first method is to click and drag the red vertical lines that appear in the diagrams. On releasing the mouse button the new factor level will appear in red together the corresponding expected yield calculated using the new set of current factor levels. The second method is to click on a bracketed factor level you wish to change. A dialog box appears via which the new level may be entered. As an exercise you should investigate the optimum yield predicted by the model if new safety regulations were to restrict the operating temperature for the process to a maximum of 240 °C – the author obtained 98.6 kg.

Before considering a 2^3 factorial experiment the opportunity will be taken to introduce further relevant displays. First consider the annotated cube plot for reactor 1 shown in Figure 8.36. Consider the temperature and pressure axes as the X and Y axes respectively, and a Z axis rising vertically out of the plane. The yields of 90 and 92 for the FLC of temperature 200 °C and pressure 4 bar may be represented by the points (200, 4, 90) and (200, 4, 92) relative to the X, Y and Z axes. (The reader unfamiliar with coordinates in three dimensions may imagine two points in space at the tips of rods that rise vertically from the bottom left-hand corner of the square. The first would be at height 90 above the square, the second at height 92.) All eight yield values plotted in this way provide a three-dimensional scatterplot, available in Minitab via **Graph > 3D Scatterplot > Simple. . . .** The dialog for the creation of such a plot is shown in Figure 8.37.

Note how in the dialog the variables Temperature, Pressure and Yield have been specified as the **X variable:**, **Y variable:** and **Z variable:** respectively for consistency with Figure 8.36 and the associated discussion. Under **Data View. . .** both **Symbols** and **Project lines** were checked. The plot is shown in Figure 8.38. The reader is advised to scrutinize the data for
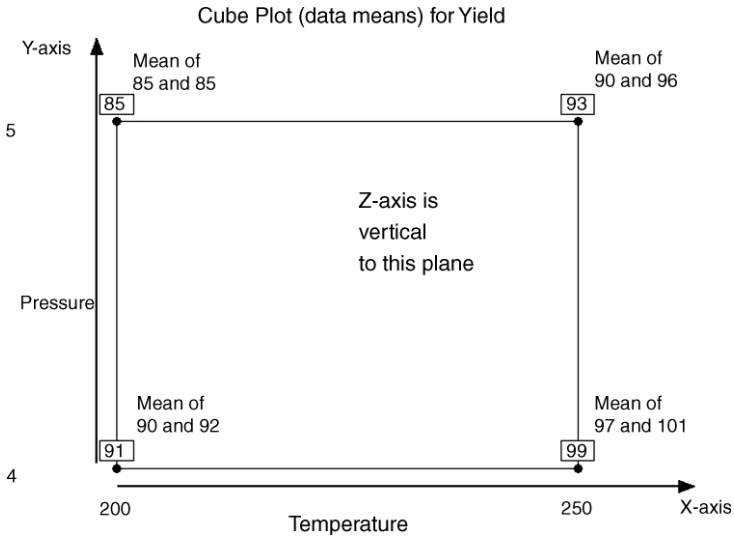
Figure 8.36   Annotated cube plot for reactor 1.

reactor 1 in worksheet Reactor1.MTW and to relate the yields to the points in the display – for example, with temperature 200 °C and pressure 5 bar, both the observed yields were 85 kg, hence the single point at the tip of the rod at the back left of the cuboid.

The fitted model equation may be used to compute an expected yield for any temperature–pressure combination within the design space. One may imagine a forest of vertical rods, like the four in the scatterplot, with points at their tips corresponding to the calculated expected yield. For example, with temperature 225 °C and pressure 4.5 bar, i.e. with $X1 = 0$ and $X2 = 0$, the predicted yield is 92 kg. The points representing the yields form the response surface predicted by the model. Minitab enables this surface to be viewed in two ways.
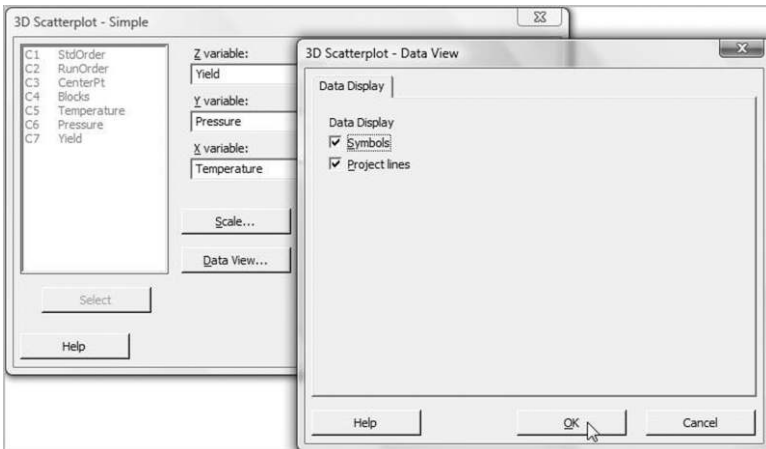

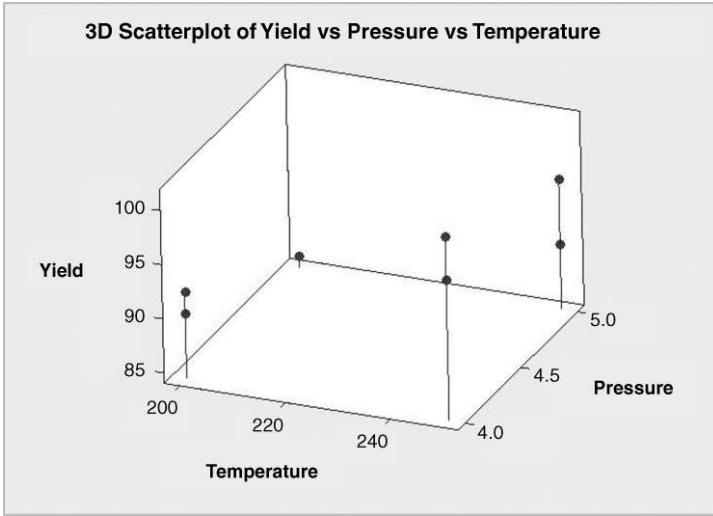**Figure 8.37**   Dialog for 3D Scatterplot.

Figure 8.38   Three-dimensional scatterplot of reactor 1 data.

One may use **Stat > DOE > Factorial > Contour/Surface Plots. . ..** In order to produce the displays in Figures 8.39 and 8.40 check both **Contour plot** and **Surface plot**. Click on **Setup. . .** for both and ensure that the column containing the response Yield is selected in both cases. The surface plot created using the default ordering of the factors is shown in Figure 8.39. The reader is invited to view the plot obtained by reversing the order of the two factors.

The response surface in the case of rector 1, where there is no interaction term, is a plane passing through the points corresponding to the mean yields at the four corners of the rectangle in Figure 8.36. The contour plot, with anotation, is shown in Figure 8.40. The contour lines in
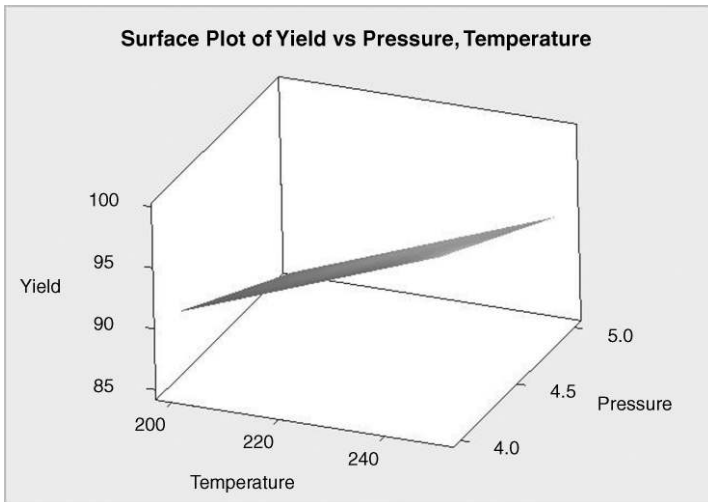

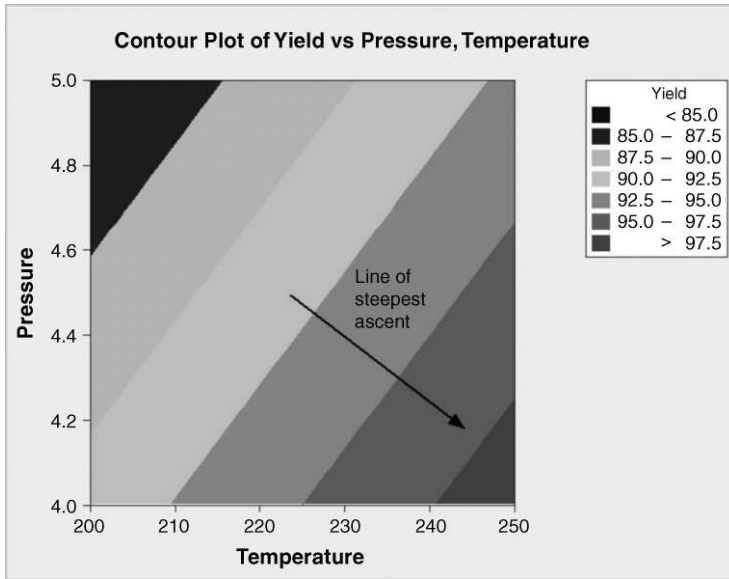**Figure 8.39**   Surface plot of reactor 1 model.

Figure 8.40   Contour plot of reactor 1 model.

the case of a plane are parallel straight lines. The line of steepest ascent is at right angles to the contour lines. The model indicates that it might be worth carrying out further experimentation in the direction of the line of steepest ascent, i.e. at temperatures above 250 °C and pressures below 4 bar, given that the aim of the project was to increase yield. The model for reactor 2 includes a nonzero interaction term. The response surface contour plot, obtained using **Contour lines** instead of **Area** via the **Contours...** button during setup, is shown in Figure 8.41. The response surface is not a plane since the contour lines are curved.

8.2.3   Examples of 2^3 and 2^4 experiments, the use of Pareto and normal probability plots of effects

As a first example of a 2^3 experiment we will consider data given by Iman and Conover (1989, p. 676) from an experiment, carried out by a large department store, as part of a project to improve on the collection of delinquent accounts. The store selected, at random, 12 delinquent accounts that had previously been in arrears and 12 delinquent accounts that had not previously been in arrears. Half of each of these 12 sets was sent only a second notice; the other half was sent a second notice accompanied by a strongly worded letter. These billings were divided again so that half contained a return envelope and half contained a prepaid return envelope. The store recorded the percentage paid on each of these 24 accounts over the 30-day period following sending out of the second notices. The data, reproduced by permission of the authors, are displayed in Table 8.14 and stored, with the design, in the worksheet Accounts.MTW.

Use of **Stat > DOE > Factorial > Analyze Factorial Design. . .** yields the Session window output shown in Panel 8.12. With the three factors there are three main effects, three first-order interactions (interactions between pairs of factors) and one second-order interaction (an interaction between three factors). In terms of the P-values given, none of the

Figure 8.41   Contour plot of reactor 2 model.
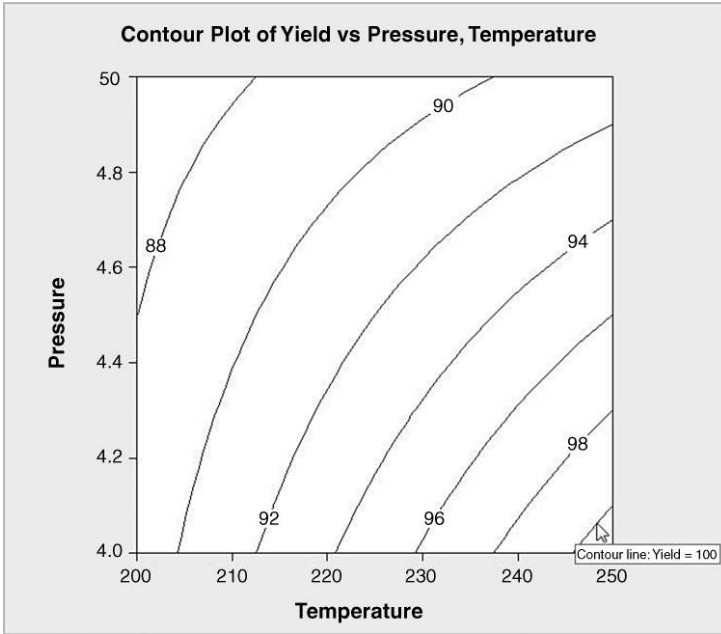
interaction effects differs significantly from zero. The main effect of billing and the main effect of previous arrears are significant at the 5% level; the main effect of postage is bordering on being significant at this level. The main effects and interaction plots are displayed in Figures 8.42 and 8.43, respectively.
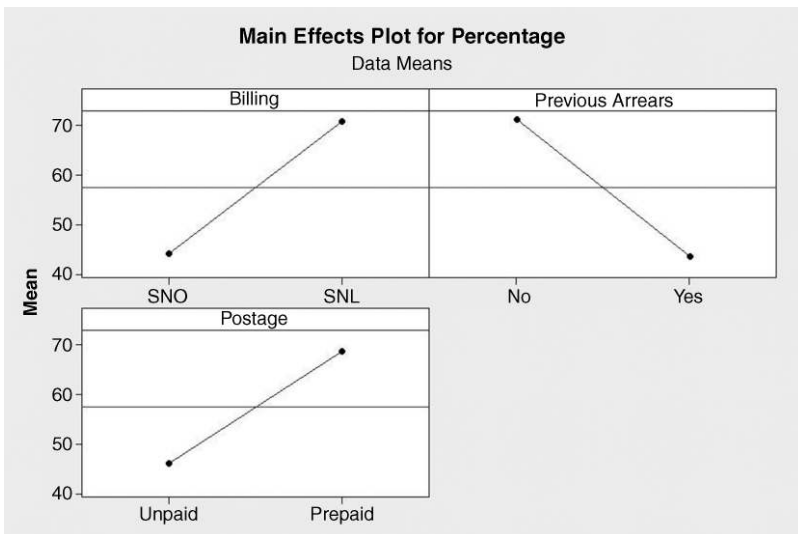

Figure 8.42   Main effects plot for accounts experiment.

Table 8.14 Data for accounts experiment.

Percentage of outstanding arrears paid off within 30 days of second billing notice being sent out		Account previously in arrears			
		No		Yes	
		Postage		Postage	
		Unpaid	Prepaid	Unpaid	Prepaid
Billing	Second	80	100	5	10
	Notice	20	40	0	60
	Only (SNO)	50	60	25	80
	Second	100	100	25	50
	Notice with	50	100	40	100
	Letter (SNL)	80	75	80	50

As there are no significant interactions the main effects plot may be interpreted at face value. Enclosure of a strongly worded letter had a beneficial effect on the response – the higher the proportion of a delinquent account paid off the better from the store's point of view. Customers with a record of previous arrears paid off a lower percentage on average than those with no such record. There is slight evidence that the provision of a prepaid envelope leads to the payment of a higher proportion of the arrears.

The nonsignificant interactions are reflected in the pairs of near parallel lines in the interaction plots displayed in Figure 8.43. The central plot in the top row of the matrix has been annotated with two vertical arrows. These indicate the similar change in the average percentage of arrears paid off, attributable to the inclusion of a strongly worded letter with the second notice, for both customers with no previous history of arrears and customers with such a history.

The next example to be considered is to be found in Box *et al.* (2005, p. 183). Three factors of interest for a pilot chemical plant were temperature, with levels 160 °C and 180 °C, concentration, with levels 20% and 40% and catalyst supplier, with levels A and B. The

```
Factorial Fit: Percentage versus Billing, Previous Arrears, Postage

Estimated Effects and Coefficients for Percentage (coded units)

Term                                Effect   Coef  SE Coef      T      P
Constant                                    57.50    5.492  10.47  0.000
Billing                              26.67  13.33    5.492   2.43  0.027
Previous Arrears                    -27.50 -13.75    5.492  -2.50  0.024
Postage                              22.50  11.25    5.492   2.05  0.057
Billing*Previous Arrears              0.83   0.42    5.492   0.08  0.940
Billing*Postage                      -5.83  -2.92    5.492  -0.53  0.603
Previous Arrears*Postage              6.67   3.33    5.492   0.61  0.552
Billing*Previous Arrears*Postage     -5.00  -2.50    5.492  -0.46  0.655


S = 26.9065     PRESS = 26062.5
R-Sq = 51.84%   R-Sq(pred) = 0.00%   R-Sq(adj) = 30.76%
```

Panel 8.12   ANOVA for 2^3 billing factorial experiment.
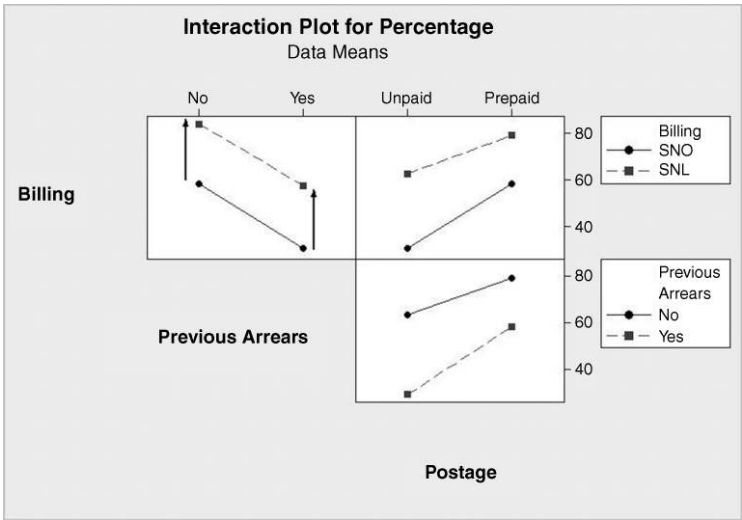
Figure 8.43   Interaction plots for accounts experiment.

response considered was yield (g) and two replications were used. We will work through the steps involved in designing the experiment and analysing the data had it been done using Minitab.

Use of **Stat** > **DOE** > **Factorial** > **Create Factorial Design. . .** offers the option to **Display Available Designs. . ..** With the default **Type of Design** 2-level factorial (default generators) selected, clicking on the button yields the display in Figure 8.44. For three factors, two designs are listed. The full 2^3 factorial experiment requires $2^3 = 8$ runs. A run is simply a factor–level combination. (The reader may find the term 'run' a little confusing. For example,

Figure 8.44   Designs available in Minitab for factors each with two levels.

in the 2^2 factorial reactor experiments considered earlier, eight yields were obtained in each. Many would say that eight runs were performed with each reactor during the experimentation. Minitab refers to four runs being replicated twice in the two experiments.) The design involving a half fraction of the full 2^3 factorial experiment requires 4 runs. The roman numbers on rectangles give the resolutions of the corresponding experimental designs. Fractional factoria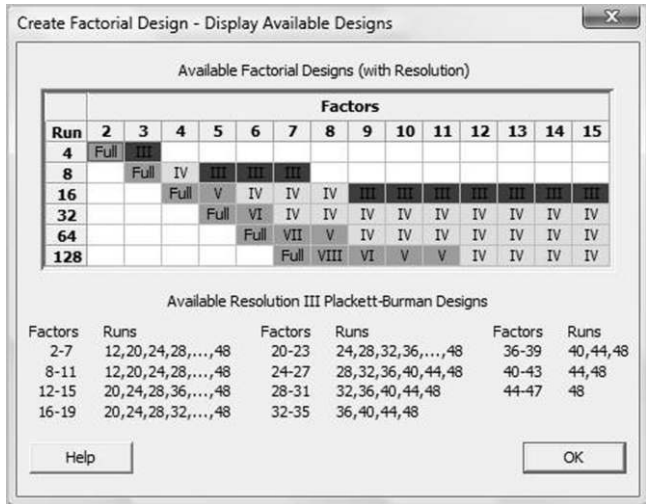l experiments and the concept of resolution will be introduced later in the chapter. Note that up to 15 factors may be used. (Plackett–Burman experimental designs will not be considered in this book.)

Having clicked on **OK**, begin the dialog by setting **Number of factors:** to 3. Then click on **Designs. . .**, highlight **Full factorial**, set **Number of replicates for coner points:** to 2 and accept the defaults of no centre points and a single block. Next the names and levels of the factors must be specified under **Factors. . .**, noting that temperature and concentration should be specified as numeric while catalyst supplier should be specified as text. The worksheet Pilot. MTW contains the data, reproduced by permission of John Wiley & Sons, Inc., New York.

The main effects plot shown in Figure 8.45 suggests that, of the three factors involved in the experiment, temperature is the major player in terms of its leverage in influencing yield. However, one must beware the temptation to claim that concentration and catalyst supplier are unimportant factors without considering interaction effects.

The major feature of the interaction plots shown in Figure 8.46 is the apparent temperature–catalyst interaction. The two versions of this specific interaction plot appear in the top right and bottom left of the matrix of plots – in the first the line segments are markedly nonparallel and in the second the line segments actually intersect.

The cube plot (a 'real' cube in the case of three factors!) is shown in Figure 8.47. One can relate features of the cube to the two earlier plots. The means of the two yields obtained for each FLC on the left face of the cube (all at temperature 160 °C) are generally much lower than the means of the two yields obtained for each FLC on the right face of the cube (all at
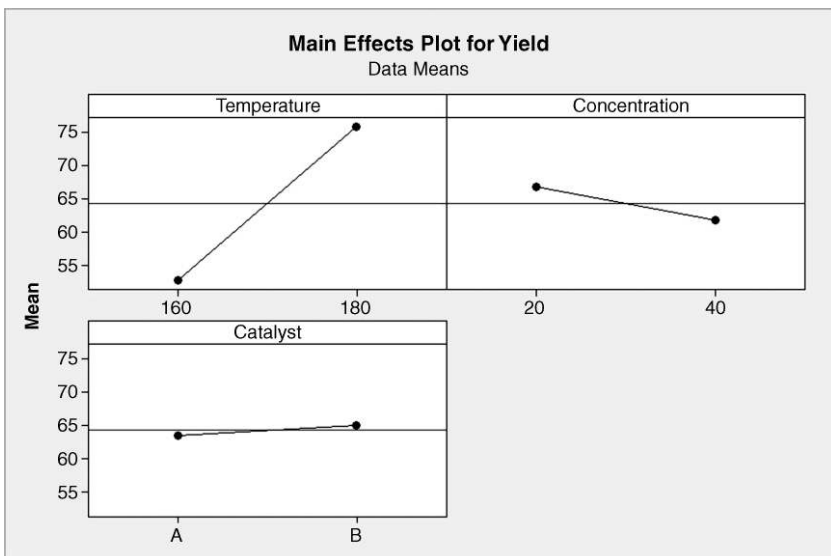

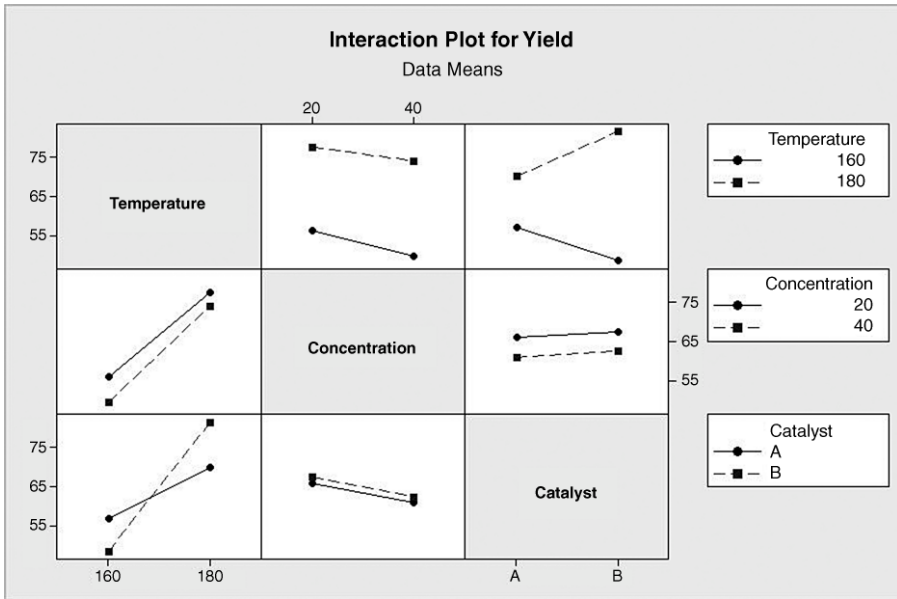**Figure 8.45**   Main effects plot for pilot chemical plant experiment.

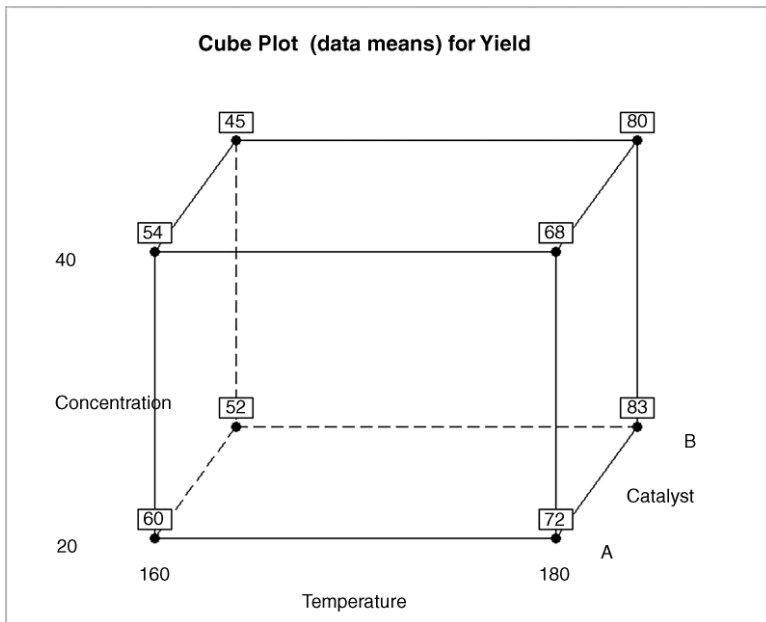**Figure 8.46**   Interaction plots for pilot chemical plant experiment.

Figure 8.47   Cube plot for pilot chemical plant experiment.

Factorial Fit: Yield versus Temperature, Concentration, Catalyst

```
Estimated Effects and Coefficients for Yield (coded units)

Term                                Effect    Coef  SE Coef       T      P
Constant                                    64.250   0.7071   90.86  0.000
Temperature                         23.000  11.500   0.7071   16.26  0.000
Concentration                       -5.000  -2.500   0.7071   -3.54  0.008
Catalyst                             1.500   0.750   0.7071    1.06  0.320
Temperature*Concentration            1.500   0.750   0.7071    1.06  0.320
Temperature*Catalyst                10.000   5.000   0.7071    7.07  0.000
Concentration*Catalyst              -0.000  -0.000   0.7071   -0.00  1.000
Temperature*Concentration*Catalyst   0.500   0.250   0.7071    0.35  0.733


S = 2.82843   R-Sq = 97.63%   R-Sq(adj) = 95.55%
```

Panel 8.13   Session window output for 2^3 pilot chemical plant experiment.

temperature 180 °C). This corresponds to the apparent major temperature main effect. The apparent temperature–catalyst interaction is indicated by the fact that changing catalyst from A to B (moving from front to back of cube) at the lower temperature leads to *decreased* yields, while on changing catalyst from A to B at the higher temperature leads to *increased* yields.

Part of the Session window output from using **Stat** > **DOE** > **Factorial** > **Analyze Factorial Design. . .** is shown in Panel 8.13. Scrutiny of the *P*-values provides evidence of a temperature–catalyst interaction and of a concentration main effect. The temperature main effect is also flagged as significant, but Box *et al.* (2005, p. 185) comment that 'the main effect of a factor should be *individually* interpreted only if there is no evidence that the factor interacts with other factors'. The main conclusions that may be drawn are as follows:

- The estimated effect of changing concentration from 20% to 40% is to *reduce* yield by 5 g on average (concentration effect is -5 from the Session window output in Panel 8.13), irrespective of the levels of the other factors, since there is no evidence of any interaction between concentration and either of the other two factors.

- The effects of temperature and catalyst cannot be interpreted separately because of the significant interaction between them. The interaction represents the different response to change in the level of temperature with the two catalysts.

This interaction may be highlighted in a cube plot that does not include the factor concentration. This is readily achieved by using **Setup. . .** for the cube plot and the arrow key for factor removal as shown in Figure 8.48. Note the right-hand arrow for selection of individual factors and the double-headed arrows for both selection and removal of all factors simultaneously. Having highlighted Concentration in the **Selected:** list clicking on the arrow pointing to the left removes the factor from the list. Clicking **OK**, removing the checks for the plots already done and clicking **OK** yields the 'collapsed' cube plot in Figure 8.49. Arrows and comments have been added to the plot.

Some final comments from the experts are relevant:

A result of great practical interest was the very different behaviors of the two 'catalyst types' in response to temperature. The effect was unexpected, for although obtained from two different suppliers, the catalysts were supposedly identical. Also, the yield from catalyst B at 180 °C was the highest that had been
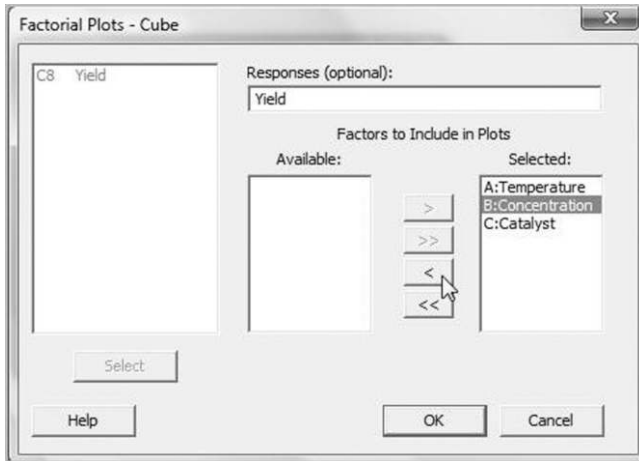
Figure 8.48   Removal and selection of factors.

seen up to that time. This finding led to a very careful study of the catalysts and catalyst suppliers in a later investigation. (Box *et al.*, 2005, p. 186)

The residual plots are satisfactory, so one can have confidence in the conclusions based on the *P*-values given in the Session window output. As an alternative to the direct use of these *P*-values to detect significant effects in the case of a 2^k factorial experiment with replication
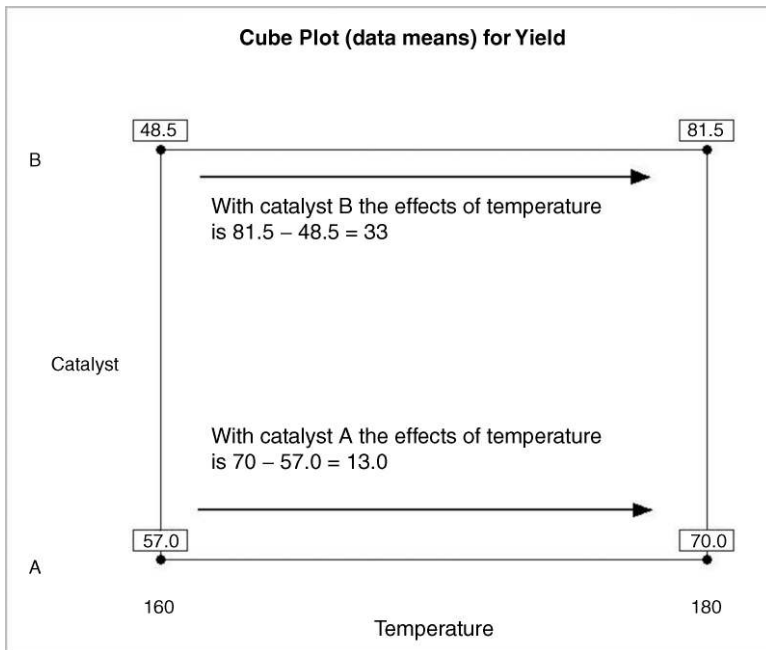
Figure 8.49   Cube plot to highlight temperature–catalyst interaction.

Normal Plot of the Standardized Effects
(response is Yield, Alpha = 0.05)
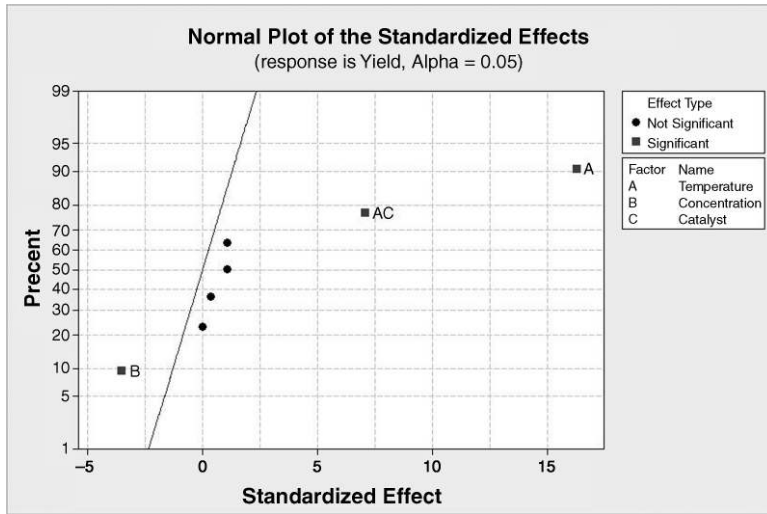
Figure 8.50   Normal probability plot of effects.

one may use a normal probability plot, a half-normal plot or a Pareto plot of the standardized effects. The standardized effects, obtained by dividing the coefficients (recall that the coefficients are half the corresponding effects) by their estimated standard errors, are in fact the t values given in Panel 8.13.

These plots are available under the **Graphs. . .** options in **Stat** > **DOE** > **Factorial** > **Analyze Factorial Design. . .**. Under **Effects Plots** one checks **Normal**, **Half Normal** or **Pareto** (or any combination) as desired. (The conclusions that may be reached from the plots are identical, so one would normally create only one of them.) Note that the default significance level **Alpha:** is 0.05. The basis of the normal plot of the effects is that, were all main effects and interaction effects in reality zero, all the estimated effects calculated from the data would constitute a random sample from a normal distribution with mean zero. Points in a normal probability plot of the effects which 'stand out from the crowd' may be taken as being associated with 'real' nonzero effects. The normal probability plot of the seven effects for the pilot chemical plant experiment is shown in Figure 8.50.

Effects that differ significantly from zero, at the chosen level of significance (5% in this case and indicated by Alpha = 0.05 in the subheading in Figure 8.50), are labelled. Use of the keys in the text boxes to the right of the plot enables the same conclusions to be reached as via the P-values in Panel 8.13.

The Pareto plot of the effects is shown in Figure 8.51. Bars that protrude through the vertical reference line correspond to effects that are significant at the chosen level. (The interested reader might check that the critical value 2.31 is the critical value for a two-tailed test using Student's t with 8 degrees of freedom.) Again the conclusions are exactly the same as before. One of the advantages of the Pareto chart is that standardized effects are ranked by absolute magnitude. The half normal plot also displays the absolute values of the standardized effects.

Table 8.15 was created by using Minitab by designing a full 2^3 factorial experiment, with one replication, no blocking, no centre points and no randomization of the run order.

Figure 8.51   Pareto plot of effects.

The default factor names A, B and C were used together with the default levels -1 and $+1$ to represent all eight FLCs. The shaded columns give the full 2^3 factorial design in standard order. The AB, AC and BC columns were calculated by direct multiplication of the entries in the appropriate columns. The ABC column may be calculated in a variety of ways, e.g. by multiplication of the entries in the AB column by those in the C column. The I column consists entirely of 1s. The final column contains the mean response for the two replicate yields obtained for each FLC. For example, with temperature 160 °C ($A = -1$), concentration 20% ($B = -1$) and catalyst A ($C = -1$) the two yields obtained were 59 g and 61 g with mean 60 g.

The A, B and C columns may be used to calculate the main effects. The AB, BC and AC columns may be used to calculate the first-order interaction effects and the ABC column may be used to calculate the second-order interaction effect. The calculations are set out in Table 8.16. The reader is invited to check some of the calculations and confirm that the results agree with the Session window output in Panel 8.13.

Table 8.15   Pilot plant data.

Pilot plant	Temp.		Conc.		Catalyst				Mean response
FLC	I	A	B	AB	C	AC	BC	ABC	Mean Y
1	1	−1	−1	1	−1	1	1	−1	60
2	1	1	−1	−1	−1	−1	1	1	72
3	1	−1	1	−1	−1	1	−1	1	54
4	1	1	1	1	−1	−1	−1	−1	68
5	1	−1	−1	1	1	−1	−1	1	52
6	1	1	−1	−1	1	1	−1	−1	83
7	1	−1	1	−1	1	−1	1	−1	45
8	1	1	1	1	1	1	1	1	80

Table 8.16 Effects for pilot plant experiment.

	Mean response at level		Effect	Coefficient
	−	+		
A	52.75	75.75	23	11.5
B	66.75	61.75	−5	−2.5
AB	63.5	65.0	1.5	0.75
C	63.5	65.0	1.5	0.75
AC	59.25	69.25	10	5
BC	64.25	64.25	0	0
ABC	64.0	64.5	0.5	0.25

We can consider the 2^3 experiment as two 2^2 experiments – one carried out by varying temperature and concentration with catalyst A only, and the other carried out by varying temperature and concentration with catalyst B only. The data for the first of these 2^2 experiments are presented in Table 8.17. The temperature–concentration interaction effect with catalyst A is given by the *AB* interaction from Table 8.17:

$$[\text{Mean yield with } AB = 1] - [\text{Mean yield with } AB = -1]$$
$$= \frac{60 + 68}{2} - \frac{72 + 54}{2} = 64 - 63 = 1.$$

The data for the second of these 2^2 experiments is presented in Table 8.18. The temperature–concentration interaction effect with catalyst B is given by *AB* interaction from Table 8.18:

Table 8.17 Pilot plant data with catalyst A.

FLC	Temp. A	Conc. B	AB	Mean response Mean Y
1	− 1	− 1	1	60
2	1	− 1	− 1	72
3	− 1	1	− 1	54
4	1	1	1	68

Table 8.18 Pilot plant data with catalyst B.

FLC	Temp. A	Conc. B	AB	Mean response Mean Y
5	−1	−1	1	52
6	1	−1	−1	83
7	−1	1	−1	45
8	1	1	1	80

$$[\text{Mean yield with } AB = 1] - [\text{Mean yield with } AB = -1]$$

$$= \frac{52 + 80}{2} - \frac{83 + 45}{2} = 66 - 64 = 2.$$

We can sum up by stating that:

- the AB interaction for $C = -1$ is 1, i.e. the temperature–concentration interaction effect with catalyst A is 1;

- the AB interaction for $C = 1$ is 2, i.e. the temperature–concentration interaction effect with catalyst B is 2.

The AB interaction effect for the full 2^3 experiment is the mean of the AB interaction for $C = -1$ and the AB interaction for $C = 1$, i.e. $(2 + 1)/2 = 1.5$. The ABC interaction effect measures the extent to which the AB interaction effects differ for the two levels of factor C. It is given by half the difference between the two AB interaction effects, i.e. by $(2 - 1)/2 = 0.5$.

A final comment before leaving this example is that the coefficient in the Session window output labelled Constant is the overall mean for the experiment. One can think of it as being generated by the column labelled I in Table 8.15. There are no negative values in the column so it corresponds to finding the mean of all 8 values in the mean response column. Familiarity with the notation introduced in Table 8.15 is vital for understanding of topics in the next section.

It is also worth examining the remainder of the Session window output. The ANOVA table is shown in the Panel 8.14. The P-value of 0.000 for main effects indicates that there is very strong evidence that the null hypothesis that all main effects are zero would be rejected. Similarly, the P-value of 0.001 for two-way (first order) interactions indicates that there is strong evidence that the null hypothesis that all interactions involving two factors are zero would be rejected. Finally, the P-value of 0.733 for three-way (second order) interactions means that there is no evidence that the ABC interaction differs from zero. The ANOVA provides a global test of significance for the various categories of effect. The P-values in the previous section of the output discussed earlier enable one to identify the specific important main effects and interaction effects.

The coefficients in the first section of the Session window output in Panel 8.13, which are half of the effects, can be used to create a model with coded variables A, B and C representing the factors and Y representing yield as follows:

$$Y = 64.25 + 11.50A - 2.50B + 0.75C + 0.75AB + 5.00AC + 0.25ABC.$$

With the process run at temperature $160\,^{\circ}\text{C}$ ($A = -1$), concentration 20% ($B = -1$) and with catalyst A ($C = -1$) the model predicts yield

```
Analysis of Variance for Yield (coded units)

Source               DF   Seq SS   Adj SS   Adj MS      F     P
Main Effects          3  2225.00  2225.00  741.667  92.71  0.000
2-Way Interactions    3   409.00   409.00  136.333  17.04  0.001
3-Way Interactions    1     1.00     1.00    1.000   0.12  0.733
Residual Error        8    64.00    64.00    8.000
  Pure Error          8    64.00    64.00    8.000
Total                15  2699.00
```

Panel 8.14   ANOVA for 2^3 pilot chemical plant experiment.

$$Y = 64.25 + 11.50(-1) - 2.50(-1) + 0.75(-1)$$
$$+ 0.75(-1)(-1) + 5.00(-1)(-1) + 0.25(-1)(-1)(-1)$$
$$= 64.25 - 11.5 + 2.5 - 0.75 + 0.75 + 5.00 - 0.25 = 60.$$

This predicted yield is the fit and is simply the mean yield obtained in the experiment with the FLC temperature 160 °C, concentration 20% and catalyst A.

A second set of coefficients is provided beneath the ANOVA table in the Session window output. These coefficients enable the model equation to be written down in terms of the uncoded variables temperature, concentration and catalyst.

The final section of the Session window output is shown in Panel 8.15. It lists the symbol I, corresponding to estimation of the overall mean, the main effects and the interaction effects. The fact that each is listed in a separate row is indicative of a full factorial experiment and that each effect may be estimated from the data. Alias structure is important in both the discussion of fractional factorial experiments and the concept of resolution in the next section.

A final example is given of a full 2^4 factorial experiment. It is taken from p. 283 of *Experimental Design with Applications in Management, Engineering, and the Sciences,* 1st edition, by Berger and Maurer, © 2002, and reprinted with permission of Brooks/Cole, a division of Thomson Learning: www.thomsonrights.com. As the title indicates, it includes nonmanufacturing applications. Table 8.19 gives numbers of coupons redeemed out of 1000 issued to each one of 16 groups of 1000 customers. The effects on this response of the factors customer willingness to use (CWTU), ease of use, value and product type were of interest. The data are available in Coupons.MTW.

With two factors, the cube plot was two-dimensional, as was the design space; with three factors, the cube plot was three-dimensional, as was the design space. The cube plot of the data in Table 8.19 is displayed in Figure 8.52. Minitab displays two cubes – one for the low level of

```
Alias Structure
I
Temperature
Concentration
Catalyst
Temperature*Concentration
Temperature*Catalyst
Concentration*Catalyst
Temperature*Concentration*Catalyst
```

Panel 8.15   Alias structure for pilot plant experiment.

Table 8.19   Data for 2^4 factorial experiment.

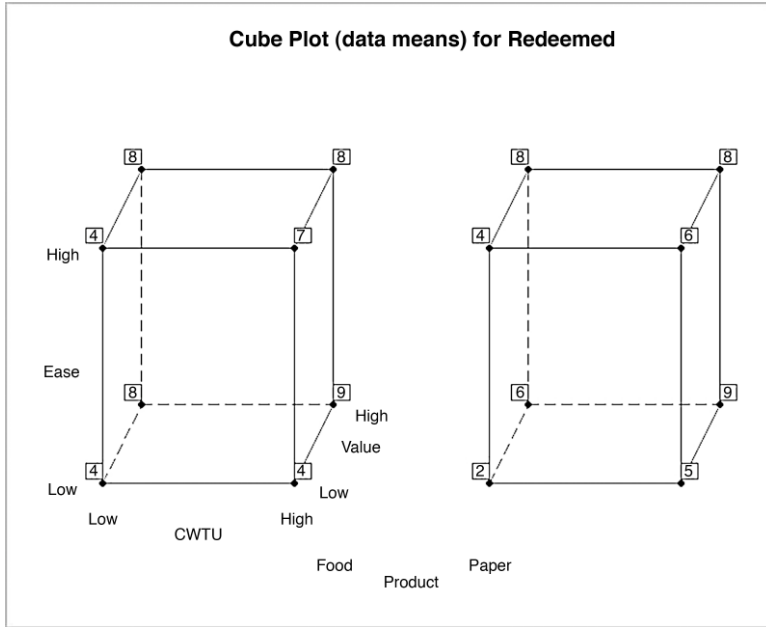| CWTU	Ease of Use	Value			
		Low		High	
		Product type		Product type	
		Food	Paper	Food	Paper
Low	Low	4	2	8	6
Low	High	4	4	8	8
High	Low	4	5	9	9
High	High	7	6	8	8

Figure 8.52   Cube plot for 2^4 factorial experiment.

product type and the other for the high level (the hypercube in four dimensions has been represented as two cubes in three dimensions!).

Part of the Session window output from **Analyze Factorial Design. . .** is displayed in Panel 8.16. The missing-value symbols indicate that no P-values can be calculated in this situation. The half normal probability plot of the standardized effects, with the default significance level $\alpha = 0.10$ selected is displayed in Figure 8.53. Thus only the main effect

```
Analysis of Variance for Redeemed (coded units)

Source                      DF   Seq SS   Adj SS   Adj MS   F   P
Main Effects                 4  61.2500  61.2500  15.3125  *   *
  CWTU                       1   9.0000   9.0000   9.0000  *   *
  Ease                       1   2.2500   2.2500   2.2500  *   *
  Value                      1  49.0000  49.0000  49.0000  *   *
  Product                    1   1.0000   1.0000   1.0000  *   *
2-Way Interactions           6   4.7500   4.7500   0.7917  *   *
  CWTU*Ease                  1   0.2500   0.2500   0.2500  *   *
  CWTU*Value                 1   1.0000   1.0000   1.0000  *   *
  CWTU*Product               1   1.0000   1.0000   1.0000  *   *
  Ease*Value                 1   2.2500   2.2500   2.2500  *   *
  Ease*Product               1   0.2500   0.2500   0.2500  *   *
  Value*Product              1   0.0000   0.0000   0.0000  *   *
3-Way Interactions           4   4.7500   4.7500   1.1875  *   *
  CWTU*Ease*Value            1   2.2500   2.2500   2.2500  *   *
  CWTU*Ease*Product          1   2.2500   2.2500   2.2500  *   *
  CWTU*Value*Product         1   0.0000   0.0000   0.0000  *   *
  Ease*Value*Product         1   0.2500   0.2500   0.2500  *   *
4-Way Interactions           1   0.2500   0.2500   0.2500  *   *
  CWTU*Ease*Value*Product    1   0.2500   0.2500   0.2500  *   *
Residual Error               0       *        *        *
Total                       15  71.0000
```

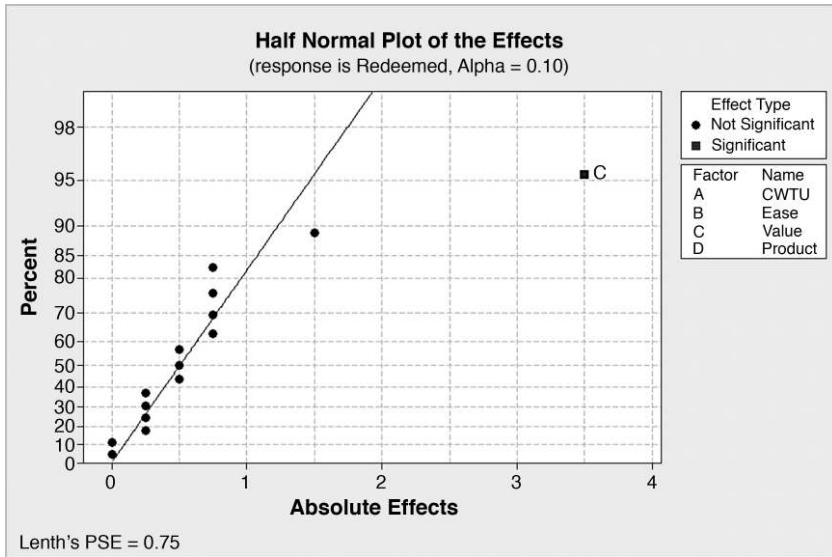**Panel 8.16**   ANOVA for coupon experiment.

Figure 8.53   Normal probability plot of effects.

of the factor labelled C, i.e. of the value factor, appears to be significantly different from zero. (Lenth's PSE, given in the bottom corner of the plot, refers to the technical detail of the method used to determine significant effects when there is no replication – see Lenth, 1989.) The reader is invited to create a main effects plot and to interpret the main effect of value.

8.3   Fractional factorial experiments in the 2^{k-p} series

8.3.1   Introduction to fractional factorial experiments, confounding and resolution

In order to introduce this topic part of a data set given in Box *et al.* (2005, p. 216) will be used. The data are reproduced by permission of John Wiley & Sons, Inc., New York. However, the context presented here was developed by the author for use in introducing the topic of fractional factorial experimental designs on training courses.

Consider a student of DOE who was asked by his tutor to design and execute a factorial experiment. The student decided to investigate the effect of the factors seat height (A, levels 26 and 30 inches) and generator state (B, levels off and on) on the response, time (Y, seconds) to travel a fixed distance up an inclined section of a road. The design he chose was a 2^2 factorial with two replications.

As no software was to be involved in the data analysis the student had prepared Table 8.20 in advance in order to display a summary of the results and as an aid to calculate the effects. The shaded columns indicate the coded levels of the factors; the four shaded rows indicating the four factor–level combinations to be used (replicated twice) in the experiment.

On showing the table to his tutor prior to carrying out the experiment, the tutor remarked that, from experience, he thought it unlikely that there would be an interaction effect between seat height and generator state. The response from the student was to ask if the AB column

Table 8.20 Pro forma for summary of bicycle experiment data.

FLC	Seat height A	Generator state B	AB	Mean time Mean Y
1	−1	−1	1	
2	1	−1	−1	
3	−1	1	−1	
4	1	1	1	

could therefore be used for the levels of a third factor. The answer was affirmative, so the student left the meeting with the revised version of the table shown in Table 8.21. He decided to allocate the factor tyre pressure (C, levels 40 and 55 psi) to the column previously labelled AB and now labelled $C=AB$. This column has now also been shaded in order to show that the levels of the three factors A, B and C to be used in the experiment are indicated by the four shaded rows.

Comparison of this table with Table 8.15 indicates that the four FLCs here comprise *half* the set of FLCs for a full 2^3 factorial experiment. Thus the student's modified proposal was to carry out what is known as a *half fraction*, with generator $C=AB$, of the full 2^3 factorial experiment. Generators are used in the creation of fractional factorial designs and also to determine their statistical properties. The interested reader will find further details in Montgomery (2009, pp. 587–594).

Half of 2^3 is $2^3/2 = 2^3/2^1 = 2^{3-1}$. Thus the design is referred to as the 2^{3-1} fractional factorial experiment with generator $C=AB$. The family of such fractional factorial designs is designated by 2^{k-p}, so, for example, a 2^{5-2} fractional factorial experiment would be a one-quarter fraction of a full 2^5 experiment since $2^{5-2} = 2^5/2^2 = 2^5/4$. A 2^{5-2} fractional factorial experiment would involve $2^5/2^2 = 32/4 = 8$ FLCs. Let us now turn to Minitab for the design and analysis of the student's bicycle experiment.

Once again use of **Stat > DOE > Factorial > Create Factorial Design. . .**, with the option **2-level factorial (default generators)** selected, is required. **Number of factors:** is specified as 3. Under **Designs. . .** the 1/2 fraction is selected with **Number of replicates for corner points:** 2 and with the defaults of no centre points and a single block The number of Runs or FLCs is four; note that Minitab represents 2^{3-1} as 2**(3-1). Under **Factors. . .** the factors (two Numeric and one Text in this case) and their levels are specified. The defaults were accepted under both **Options. . .** (i.e to **Randomize runs** and to **Store design in worksheet**) and **Results. . .**. The resultant worksheet should be augmented with a column showing the response values, i.e. the times, and a column in which the student could note any unusual happening or information that might prove relevant to the analysis of the data from the

Table 8.21 Revised pro forma for bicycle experiment data.

FLC	Seat height A	Generator state B	Tyre pressure C = AB	Mean time Mean Y
1	− 1	− 1	1	
2	1	− 1	− 1	
3	− 1	1	− 1	
4	1	1	1	

C1	C2	C3	C4	C5	C6-T	C7	C8	C9-T
StdOrder	RunOrder	CenterPt	Blocks	Seat Height	Generator State	Tyre Pressure	Time	Remarks
6	1	1	1	30	Off	40	41	Snow on road
8	2	1	1	30	On	55	41	
5	3	1	1	26	Off	55	50	
3	4	1	1	26	On	40	54	
2	5	1	1	30	Off	40	43	Melting snow
4	6	1	1	30	On	55	44	
1	7	1	1	26	Off	55	48	
7	8	1	1	26	On	40	60	Snow

Figure 8.54 Worksheet for bicycle experiment.

experiment and to the project. As ever, the worksheet should be saved both for storage of the data and to facilitate analysis once the experimental work has been completed (see Figure 8.54, where the response data and comments have been added; the data are provided in Bicycle.MTW).

The information that appears in the Session window, on creation of the experimental design, is displayed in Panel 8.17. Base Design: 3, 4 indicates that three factors are involved and that four FLCs are employed. The design has resolution III, which means that 'no main effects are aliased with any other main effect, but main effects are aliased with two-factor interactions and some two-factor interactions may be aliased with each other' (Montgomery, 2009, p. 589). The student will make a total of eight experimental runs. (Minitab is inconsistent in its use of the term 'runs' – in the dialog for the creation of the design, 'runs' refers to the number of FLCs.) Two runs will be made with each FLC. A half fraction of the full factorial is being used and the eight runs will constitute a single block. No centre points are to be used.

The price paid for using this fractional factorial is that some main effects are confounded with two-way interactions (*A* with *BC*, *B* with *AC*, and *C* with *AB* – in each case the main effect and the interaction are aliases of each other), which means that if an interaction between a pair of factors exists then we will be unable to 'disentangle' it from a main effect in our analysis. This type of confounding is a feature of resolution III designs. Note also that the overall mean (*I*) is confounded with the *ABC* interaction.

The equation $C = AB$ indicates how the design may be generated. Note that this was how a factor was allocated to the third column in the discussion between student and tutor.

```
Fractional Factorial Design

Factors:  3    Base Design:        3, 4    Resolution:  III
Runs:     8    Replicates:            2    Fraction:    1/2
Blocks:   1    Center pts (total):    0

* NOTE * Some main effects are confounded with two-way interactions.

Design Generators: C = AB


Alias Structure

I + ABC

A + BC
B + AC
C + AB
```

Panel 8.17 Specification of 2^{3-1} design for bicycle experiment.

As stated above, the design considered here is of resolution III. With resolution IV designs no main effect is aliased with a two-factor interaction, but at least one two-factor interaction is aliased with another two-factor interaction. With resolution V designs no main effect or two-factor interaction is aliased with either a main effect or a two-factor interaction. The higher the resolution of an experimental design, the greater its potential for the unambiguous identification of important main effects and interaction effects. Full factorial design experiments involving replication have the capability to provide information, without any confounding, on all main effects and all interactions and may be said to have 'infinite' resolution. Resolution III designs are often used for screening experiments, carried out with the aim of identifying potentially important factors from a set of factors identified as being worth investigation. 'For most practical purposes, a resolution 5 design is excellent and a resolution 4 design may be adequate' (NIST/SEMATECH, 2005, Section 5.7).

The cube plot of the data is shown in Figure 8.55. Note the four vertices corresponding to the four FLCs used in the experiment. The mouse pointer is shown located at the vertex corresponding to the FLC with seat height 30, generator on and tyre pressure 55. The two times achieved were 41 and 44, giving the mean of 42.5 displayed. (The other vertices provide an alternative half fraction of the full 2^3 design.)

The main effects plot is shown in Figure 8.56. The major effect would appear to be that of seat height, with runs done with the seat in the high position taking around 11 seconds less on average than those runs done with the seat in the low position. Runs with the generator on took around 4 seconds longer than those with the generator off. This is not surprising as, with the generator on, some of the cyclist's energy will be required to drive the generator. Runs with the tyre pressure high took about 4 seconds less on average than those with tyre pressure low. Thus a pick-a-winner approach on the basis of this plot would suggest seat height set high, generator turned off and tyre pressure set high to achieve the shortest possible time. However, we must bear in mind that this could be totally in error if there are significant interaction effects.

Proceeding to formal analysis via **Analyze Factorial Design. . .**, it is informative to examine the default offered under **Terms. . .** as displayed in Figure 8.57. The default is to **Include terms in the model up through order:** 1, i.e. to include only terms corresponding
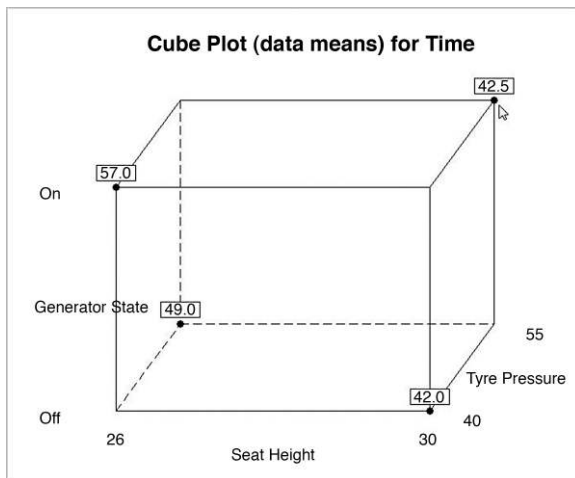

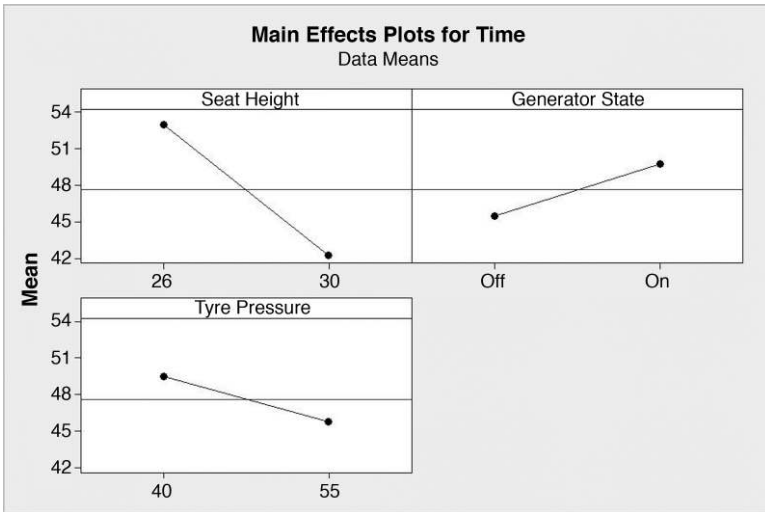**Figure 8.55** Cube plot for bicycle experiment.

Figure 8.56   Main effects plot for bicycle experiment.

to main effects. These are listed under **Selected Terms:**. The arrow keys may be used to change the selection of terms.

Part of the Session window output from the analysis is shown in Panel 8.18. If we assume that there are no significant interactions, then the analysis provides strong evidence of a seat height effect, slight evidence of a generator state effect and marginal evidence of a tyre pressure effect (at the 10% significance level). Let us decide to keep all three terms in the model and let
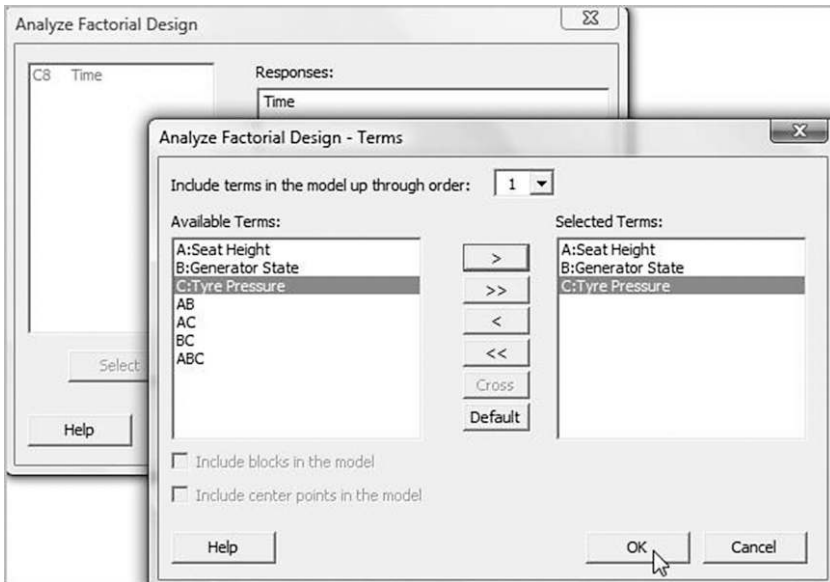

Figure 8.57   Default terms for model for bicycle experiment.

```
Factorial Fit: Time versus Seat Height, Generator State, Tyre Pressure

Estimated Effects and Coefficients for Time (coded units)

Term              Effect    Coef  SE Coef      T      P
Constant                   47.625  0.9100  52.33  0.000
Seat Height      -10.750  -5.375  0.9100  -5.91  0.004
Generator State    4.250   2.125  0.9100   2.34  0.080
Tyre Pressure     -3.750  -1.875  0.9100  -2.06  0.108
```

Panel 8.18   Session window output from analysis of bicycle experiment.

us suppose that the student would like to achieve a time of 35 seconds but would be satisfied with 40 seconds. Use of **Stat** > **DOE** > **Factorial** > **Response Optimizer. . .** with time as the (only) response, **Goal set** to Minimize, **Upper** to 40 and **Target** to 35 yields the display in Figure 8.58. This indicates that a mean time of 38.25 seconds is predicted with the FLC seat height set high at 30 in, generator turned off and tyre pressure set high at 55 psi. These levels correspond to those suggested by pick-a-winner scrutiny of the main effects plot in Figure 8.56.

The FLC indicated was not one that was employed in the experiment. Thus confirmation trials with the predicted optimal combination of factor levels are clearly advisable. (In fact the student actually carried out the full 2^3 factorial. Both times achieved with the seat height set high at 30 in, generator turned off and tyre pressure set high at 55psi were 39 seconds. Thus the prediction from the fractional factorial was good. The analysis of the full data set provided no evidence of any significant interaction effects.) The user can, of course, vary the factor levels with the response optimizer, as indicated in the previous section, in the case of seat height and tyre pressure. However, with the factor generator state being of text type, no intermediate values are possible. If the user tries to drag the red line to an intermediate position, release of the mouse button causes the factor level to 'jump' to the other of the two discrete levels. The two solid circles indicate the nature of this particular factor.

8.3.2 Case study examples

Montgomery (2009, p. 594) gives an example of a 2^{7-3} fractional factorial experiment conducted in an effort to reduce shrinkage of parts manufactured using an injection moulding process. Since $2^{7-3} = 2^7/2^3 = 128/8 = 16$, the experiment involved 16 of the 128 FLCs
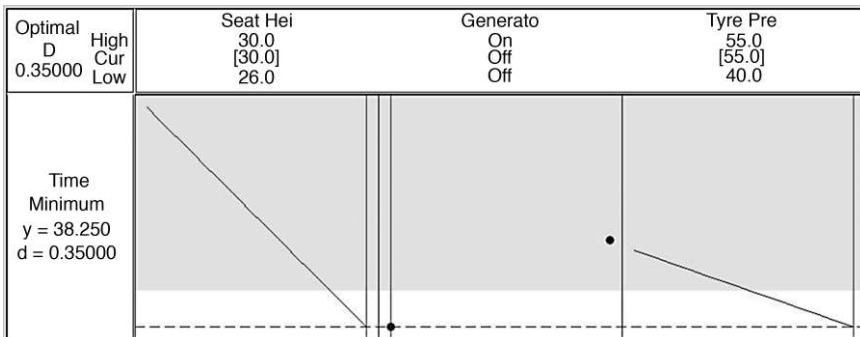

Figure 8.58   Response Optimizer for bicycle experiment.

```
Alias Structure

I + ABCE + ABFG + ACDG + ADEF + BCDF + BDEG + CEFG

A + BCE + BFG + CDG + DEF + ABCDF + ABDEG + ACEFG
B + ACE + AFG + CDF + DEG + ABCDG + ABDEF + BCEFG
C + ABE + ADG + BDF + EFG + ABCFG + ACDEF + BCDEG
D + ACG + AEF + BCF + BEG + ABCDE + ABDFG + CDEFG
E + ABC + ADF + BDG + CFG + ABEFG + ACDEG + BCDEF
F + ABG + ADE + BCD + CEG + ABCEF + ACDFG + BDEFG
G + ABF + ACD + BDE + CEF + ABCEG + ADEFG + BCDFG
AB + CE + FG + ACDF + ADEG + BCDG + BDEF + ABCEFG
AC + BE + DG + ABDF + AEFG + BCFG + CDEF + ABCDEG
AD + CG + EF + ABCF + ABEG + BCDE + BDFG + ACDEFG
AE + BC + DF + ABDG + ACFG + BEFG + CDEG + ABCDEF
AF + BG + DE + ABCD + ACEG + BCEF + CDFG + ABDEFG
AG + BF + CD + ABDE + ACEF + BCEG + DEFG + ABCDFG
BD + CF + EG + ABCG + ABEF + ACDE + ADFG + BCDEFG
ABD + ACF + AEG + BCG + BEF + CDE + DFG + ABCDEFG
```

Panel 8.19   Alias structure for a 2^{7-3} fractional factorial design.

required for a full 2^7 factorial. The experiment was a 1/8 fraction of the full factorial. The factors were mould temperature, A; screw speed, B; holding time, C; cycle time, D; moisture content, E; gate size, F; and holding pressure G. The response, Y, was shrinkage measured as a percentage, and a single replication was used. The data are available in Shrinkage.MTW and are reproduced by permission of John Wiley & Sons, Inc., New York. The alias structure of this design is displayed in Panel 8.19. The reader is invited to check this using **Stat** > **DOE** > **Factorial** > **Create Factorial Design. . .** with the option **2-level factorial (default generators)** selected and **Number of factors:** 7 specified. Under **Designs. . .** the 1/8 fraction is selected with defaults otherwise The number of runs or FLCs is 16; note that Minitab represents 2^{7-3} as $2^{**}(7\text{-}3)$. The alias structure in Panel 8.19 then appears in the Session window.

The design is of resolution IV. Main effects and interactions between sets of three factors are confounded with each other (and with higher-order interactions). Interactions between pairs of factors are confounded with each other (and with higher-order interactions). Resolution IV designs are such that 'no main effect is aliased with any other main effect *or* with any two-factor interaction, but two-factor interactions are aliased with each other' (Montgomery, 2009, p. 592).

On the basis of the annotated normal probability plot of the effects displayed in Figure 8.59, the engineers involved with the process made the tentative conclusion that the important factors were likely to be mould temperature A and screw speed B. The level of significance 0.001 was used in creating the plot. The conclusion has to be a tentative one since, on the assumption that all interactions involving three or more factors are zero, the effect labelled AB in the plot could be important because any of the AB, CE and FG interactions could be important or indeed any combination involving two or more of them. This can be confirmed by checking the alias structure in Panel 8.19 (tenth line).

The engineers agreed to set temperature at the low level and screw speed at the low level on the strength of scrutiny of the interaction plot displayed in Figure 8.60. Clearly at the low screw speed the effect of temperature appears to be negligible – in such situations the choice of level of temperature to use can be based on cost considerations.

The project team fitted a model involving only the mould temperature, the screw speed and the interaction between them. They plotted the residuals from the fitted model against the
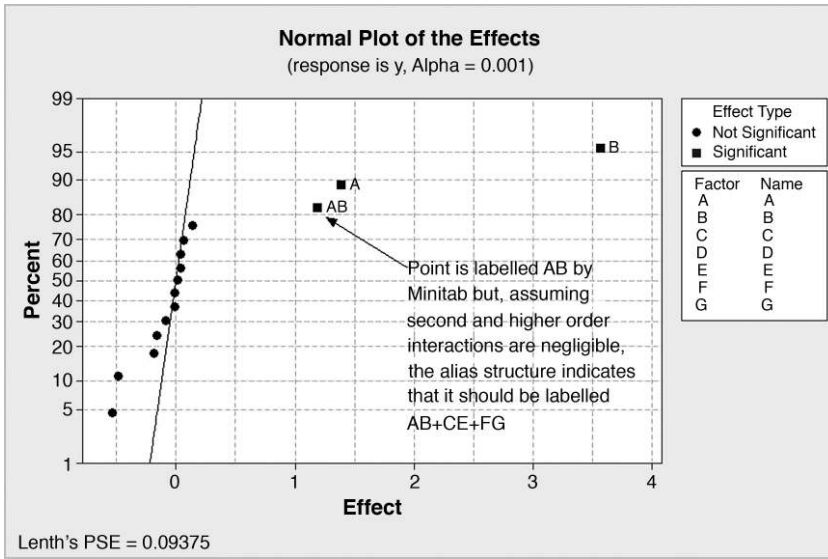
Figure 8.59   Normal probability plot of effects for injection moulding experiment.

levels of the other factors not in the model. This may be readily achieved in Minitab using **Residuals versus variables:** under **Graphs. . .** in **Analyze Factorial Design. . .** and selecting the five factors not in the model. Of particular interest was the plot of residuals against holding time, shown in Figure 8.61, as it suggests that at the low level of holding time (factor *C*) there would be less variability in shrinkage.

This led the project team to investigate running the process with low mould temperature, low screw speed and low holding time. The aim was to ascertain whether or not it could be
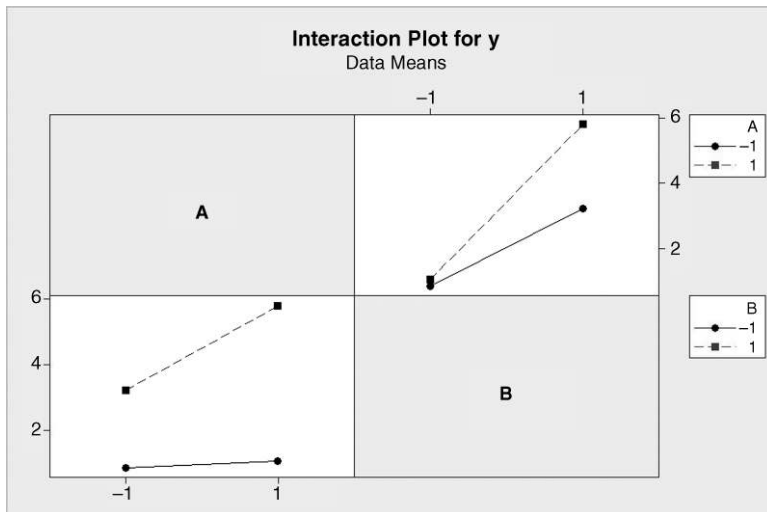

Figure 8.60   *AB* interaction plot for the injection moulding experiment.
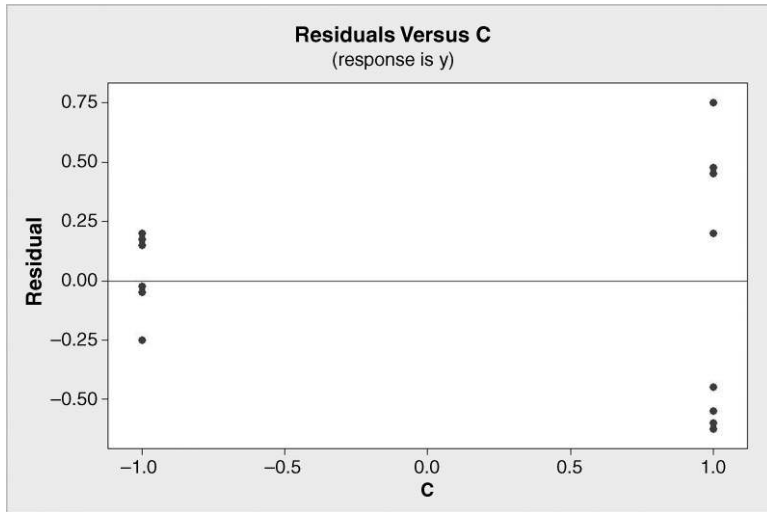
Figure 8.61   Plot of residuals versus holding time.

confirmed that factors A and B were indeed influencing the response in terms of its location and that factor C was influencing the response in terms of its variability.

An advertising agency conducted an experiment in which various forms of newspaper advertisement for a new leisure product were created. The factors and levels were: size of advertisement (A, with levels small and large); type of advertisement (B, simple and complex); colours (C, two and four); setting (D, sole and group); and newspaper style (E, compact and broadsheet). The response was the proportion (expressed as a percentage) of a sample of people who were able to remember reading the advertisement, having read through a mock-up newspaper containing it. The experiment was a 2^{5-1} design, i.e. it was a half fraction of the full 2^5 factorial and therefore involved 16 FLCs. A single replication was used. The data are available in Advertising.MTW.

On use of **Analyze Factorial Design...** to perform an analysis of the data the alias structure for the design that was obtained, which has resolution V, is shown in Panel 8.20. Main effects are confounded with third-order interactions (i.e. interactions involving four factors), and first-order interactions (i.e. interactions involving pairs of factors) are confounded with second-order interactions (i.e. interactions involving sets of three factors). This alias structure highlights the important property of resolution V designs, namely that 'no main effect or two-factor interaction is aliased with any other main effect or two-factor interaction, but two-factor interactions are aliased with three-factor interactions' (Montgomery, 2009, p. 592) Thus, if second- and higher-order interactions are negligible, as is frequently the case, then main effects and two-factor interactions can be separately estimated. The normal probability plot of the effects is shown in Figure 8.62.

The main effects of factors B and E are flagged as being potentially important, together with the first-order interaction effect CD. The nature of these effects my be gleaned from scrutiny of the main effects plot in Figure 8.63 and the interaction plot in Figure 8.64. The main effect of B may be interpreted as a reduction in recall of around 7% on changing from simple to complex type. The main effect of E may be interpreted as an increase in recall of around 10% on changing from compact to broadsheet style of newspaper.

```
Alias Structure
I + A*B*C*D*E
A + B*C*D*E
B + A*C*D*E
C + A*B*D*E
D + A*B*C*E
E + A*B*C*D
A*B + C*D*E
A*C + B*D*E
A*D + B*C*E
A*E + B*C*D
B*C + A*D*E
B*D + A*C*E
B*E + A*C*D
C*D + A*B*E
C*E + A*B*D
D*E + A*B*C
```

Panel 8.20 Alias structure for a 2^{5-1} fractional factorial design.

The *CD* interaction may be interpreted as follows: changing from two to four colours (i.e. from $C = -1$ to $C = 1$) when the advertisement is the only one on the page ($D = -1$) leads to a decrease in the recall rate, whereas changing from two to four colours (i.e. from $C = -1$ to $C = 1$) when the advertisement is one of a group on the page ($D = 1$) leads to an increase in the recall rate.

8.4 Taguchi experimental designs

Figure 8.65 shows the two possible half fractions of the full 2^3 factorial experiment. The circular symbols represent the principal half fraction of the full 2^3 factorial that was introduced
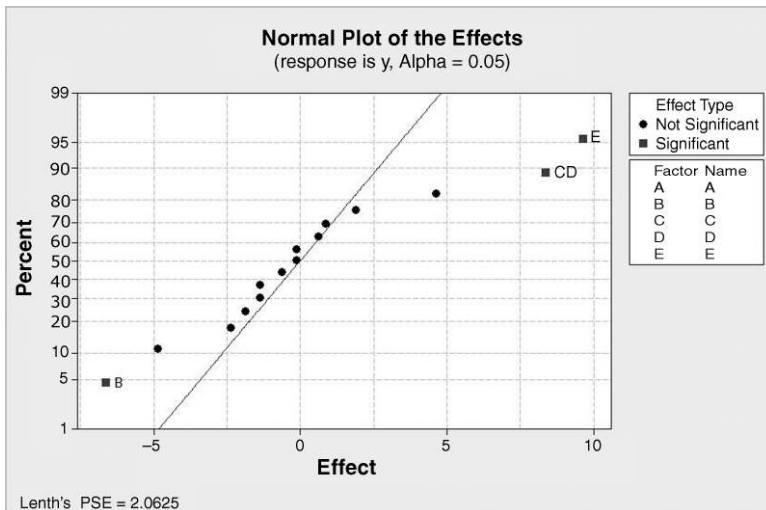

Figure 8.62 Normal probability plot of effects for advertising experiment.
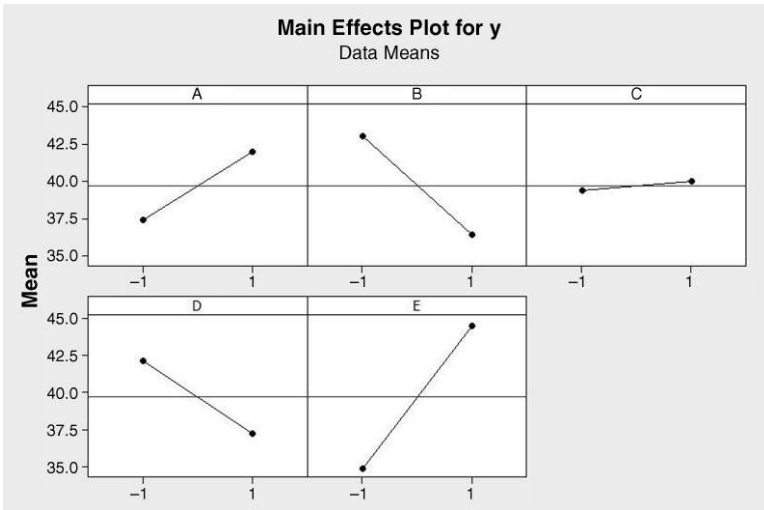
Figure 8.63   Main effects plot for advertising experiment.

earlier for the bicycle experiment – see Table 8.21 and Figure 8.55. The generator of this fraction is $C = AB$. An alternative half fraction is represented by the triangular symbols in Figure 8.65. The generator of this fraction is $C = -AB$, as the reader can readily verify from scrutiny of Table 8.23.

The numbers in the A, B and C columns define three vectors in a space of four dimensions. The inner product of two vectors is calculated by obtaining the sum of the products of corresponding components of the two vectors. For example, the inner product of A and B is

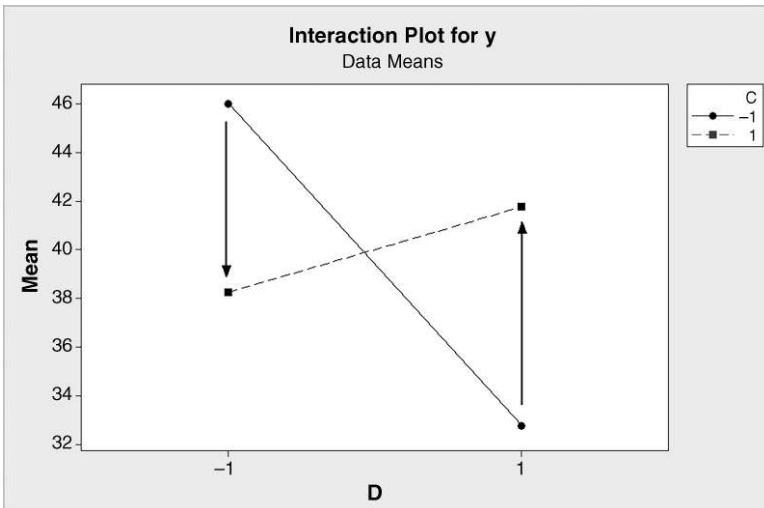$$(-1) \times (-1) + 1 \times (-1) + (-1) \times 1 + 1 \times 1 = 1 - 1 - 1 + 1 = 0.$$

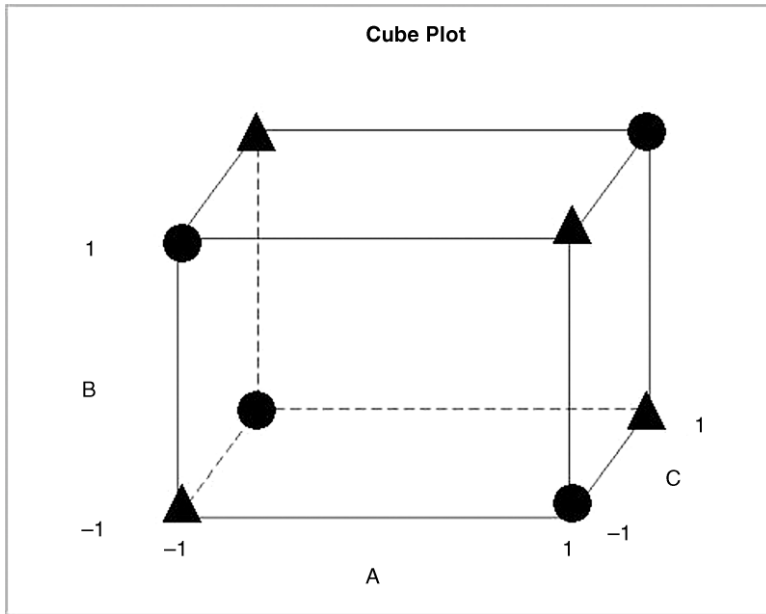**Figure 8.64**   CD interaction plot for advertising experiment.

Figure 8.65   The two possible half fractions of the full 2^3 factorial.

The reader is invited to verify that the inner product of A and C and the inner product of vectors B and C are also zero. Vectors with inner product of zero are said to be *orthogonal*. Since the A, B and C columns of factor levels for the experimental design in Table 8.23 are made up of orthogonal vectors the design constitutes an orthogonal array.

If the factor levels in this design are denoted by 1 and 2 rather than −1 and 1 then, with some rearrangement of the rows, we have the experimental design, designated *Taguchi's $L_4(2^3)$ orthogonal array*, displayed in Table 8.24. Genichi Taguchi was a Japanese engineer who did

Table 8.22   Principal half fraction of the full 2^3 experiment.

FLC	A	B	C
1	− 1	− 1	1
2	1	− 1	− 1
3	− 1	1	− 1
4	1	1	1

Table 8.23   Alternative half fraction of the full 2^3 experiment.

FLC	A	B	C
1	− 1	− 1	− 1
2	1	− 1	1
3	− 1	1	1
4	1	1	− 1

Table 8.24   Taguchi's $L_4(2^3)$ orthogonal array.

FLC	A	B	C
1	1	1	1
2	1	2	2
3	2	1	2
4	2	2	1

Figure 8.66   Linear graph for the $L_4(2^3)$ orthogonal array.

much to promote the use of designed experiments in industry and made other significant contributions to quality improvement thinking and methodology. Taguchi created linear graphs associated with each orthogonal array; that for the $L_4(2^3)$ array is displayed in Figure 8.66. The graph, reproduced by permission of the American Supplier Institute Inc., Dearborn, indicates that, if factors A, B and C are respectively assigned to columns 1, 2 and 3 of the orthogonal array, the interaction between the factors A (1) and B (2) will be aliased or confounded with the main effect of C (3). Both Kolarik (1995, pp. 892–901) and Hicks and Turner (1999, pp. 370–398) give comprehensive lists of Taguchi orthogonal arrays and their associated linear graphs. In Minitab the available Taguchi experimental designs may be accessed via **Stat > DOE > Taguchi > Create Taguchi Design...**.

Taguchi advocated the use of signal to noise ratios in the analysis of data from designed experiments. The three principal ratios are given in Box 8.5. Each corresponds to a different

1. Target (nominal is best)

$$SN_T = 10 \log_{10}\left(\frac{\bar{y}^2}{s^2}\right) \quad \text{or} \quad SN_T = -10 \log_{10} s^2$$

2. Maximisation (larger is better)

$$SN_L = -10 \log_{10}\left(\frac{1}{n}\sum_{i=1}^{n} \frac{1}{y_i^2}\right)$$

3. Minimisation (smaller is better)

$$SN_S = -10 \log_{10}\left(\frac{1}{n}\sum_{i=1}^{n} y_i^2\right)$$

Box 8.5   Taguchi signal to noise ratios.

objective. The first may be employed when the objective is to achieve a target value for the response, the second when it is desired to maximize a response, and the third when minimization of a response is required. These signal to noise ratios were developed in such a way that in all cases the larger the value of the ratio the better in terms of achieving objectives. At a symposium on Taguchi methods in 1985, J. Quinlan and colleagues from Flex Products reported on an experiment that they had carried out as part of a quality improvement project with the objective of minimization of the shrinkage of extruded thermoplastic speedometer cable casing (Quinlan, 1985, pp. 257–266). In their introduction they refer to the fact that the previous manufacturer had 'conducted much one factor at a time experimentation with high costs and disappointing results'. The 15 factors of interest in the experiment are listed in Table 8.25 together with the two levels for each that were selected by the experimenters.

In order to design a Taguchi experiment one may use **Stat > DOE > Taguchi > Create Taguchi Design. . ..** On selecting **Number of factors:** 15, Minitab offers the $L_{16}(2^{15})$ and $L_{32}(2^{15})$ orthogonal arrays as shown in Figure 8.67. The researchers chose the former design, which involves 16 FLCs. For each of these combinations 3000 feet of the casing was produced. From each length four sections were removed and the percentage shrinkage measured. The data are provided in Casing.MTW, displayed in Table 8.26 and reproduced by permission of the American Supplier Institute Inc., Dearborn, MI. With this type of experimental design Minitab offers no randomisation procedure. However, in the conduct of the experiment the various FLCs were randomized 'as much as possible'.

Before discussing the analysis of the actual experimental data we will consider hypothetical data shown in Table 8.27 for three FLCs. It would appear that FLCs II and III are equally good in that both give mean percentage shrinkage of 0.15, compared with 0.50 for combination I. For FLC I the appropriate 'smaller is better' signal to noise ratio is calculated as

Table 8.25   4 Factors and levels for speedometer cable casing experiment.

| Factor	Name	Level 1	Level 2
A	Liner OD	Existing	Changed
B	Liner die	Existing	Changed
C	Liner material	Existing	Changed
D	Liner line speed	Existing	80% of existing
E	Wire braid type	Existing	Changed
F	Braiding tension	Existing	Changed
G	Wire diameter	Smaller	Existing
H	Liner tension	Existing	More
J	Liner temperature	Ambient	Preheated
K	Costing material	Existing	Changed
L	Coating die type	Existing	Changed
M	Melt temperature	Existing	Cooler
N	Screen pack	Existing	Denser
O	Cooling method	Existing	Changed
P	Line speed	Existing	70% of existing
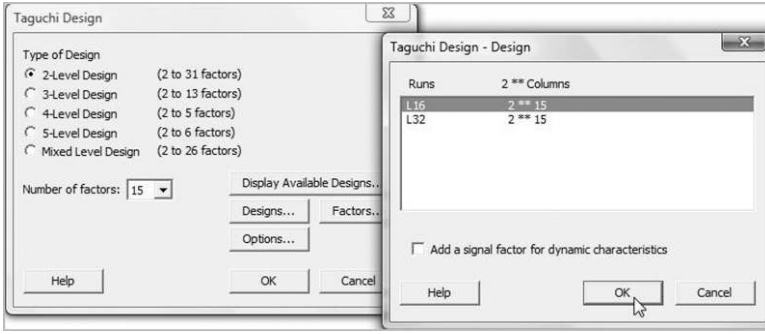
Figure 8.67 Selecting the $L_{16}(2^{15})$ orthogonal array.

$$SN_S = -10 \log_{10}\left(\frac{1}{n}\sum_{i=1}^{n} y_i^2\right)$$

$$= -10 \log_{10}\left(\frac{1}{4}\left(0.56^2 + 0.44^2 + 0.54^2 + 0.46^2\right)\right)$$

$$= -10 \log_{10}\left(\frac{1.0104}{4}\right)$$

$$= -10 \log_{10} 0.2526$$

$$= -10 \times (-0.598) = 5.98.$$

Table 8.26 Data for speedometer cable casing experiment.

A	B	C	D	E	F	G	H	J	K	L	M	N	O	P	y1	y2	y3	y4
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0.49	0.54	0.46	0.45
1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	0.55	0.60	0.57	0.58
1	1	1	2	2	2	2	1	1	1	1	2	2	2	2	0.07	0.09	0.11	0.08
1	1	1	2	2	2	2	2	2	2	2	1	1	1	1	0.16	0.16	0.19	0.19
1	2	2	1	1	2	2	1	1	2	2	1	1	2	2	0.13	0.22	0.20	0.23
1	2	2	1	1	2	2	2	2	1	1	2	2	1	1	0.16	0.17	0.13	0.12
1	2	2	2	2	1	1	1	1	2	2	2	2	1	1	0.24	0.22	0.19	0.25
1	2	2	2	2	1	1	2	2	1	1	1	1	2	2	0.13	0.19	0.19	0.19
2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	0.08	0.10	0.14	0.18
2	1	2	1	2	1	2	2	1	2	1	2	1	2	1	0.07	0.04	0.19	0.18
2	1	2	2	1	2	1	1	2	1	2	2	1	2	1	0.48	0.49	0.44	0.41
2	1	2	2	1	2	1	2	1	2	1	1	2	1	2	0.54	0.53	0.53	0.54
2	2	1	1	2	2	1	1	2	2	1	1	2	2	1	0.13	0.17	0.21	0.17
2	2	1	1	2	2	1	2	1	1	2	2	1	1	2	0.28	0.26	0.26	0.30
2	2	1	2	1	1	2	1	2	2	1	2	1	1	2	0.34	0.32	0.30	0.41
2	2	1	2	1	1	2	2	1	1	2	1	2	2	1	0.58	0.62	0.59	0.54

Table 8.27   Hypothetical shrinkage data.

FLC	Shrinkage				Statistics	
	y1	y2	y3	y4	Mean	SN_S
I	0.56	0.44	0.54	0.46	0.50	5.98
II	0.21	0.09	0.19	0.11	0.15	16.00
III	0.15	0.15	0.16	0.14	0.15	16.47

FLC III gives a slightly larger signal to noise ratio (16.47) than FLC II (16.00) since the responses for it have less variability than the responses for FLC II. Thus the 'best' FLC yields the highest signal to noise ratio.

In order to analyse the data from the actual experiment use may be made of **Stat** > **DOE** > **Taguchi** > **Analyze Taguchi Design. . .**. The dialog is displayed in Figure 8.68. **Response data are in:** y1-y4 indicates the location of the response values in the worksheet. (The reader should note that the layout of the data differs from that in the case of the factorial experiments considered earlier, where replicate responses appeared in different rows of the same column of the Minitab worksheet for analysis.) Under **Options. . .** a signal to noise ratio appropriate to the context is selected – here **Smaller is better** is appropriate. Under **Graphs. . .** and **Analysis. . .** the defaults were accepted. **Storage. . .** was used to specify that columns containing the means and signal to noise ratios should be computed for each FLC.

The main effects plot for both the mean response and the signal to noise ratio are displayed in Figures 8.69 and 8.70. The 'main players' appear to be factors E and G. However, Minitab has provided no information on the aliasing involved in this design. In fact the design is of resolution III, so main effects are confounded with first-order interactions between pairs of factors. Thus any major apparent observed effect might be due to an important interaction.

Assuming that there are no important interactions, and bearing in mind that the objective was to reduce shrinkage as much as possible, the main effects plot of means for factor E (wire braid type) indicates that level 2 (changed) should be used since it was desirable that shrinkage be as low as possible.

In scrutinizing the corresponding main effects plot of the signal to noise ratios there is no conflict. Level 2 for factor E (wire braid type) is again indicated since the signal to noise ratio is defined in such a way that we always seek to achieve the largest possible values. Similarly, both
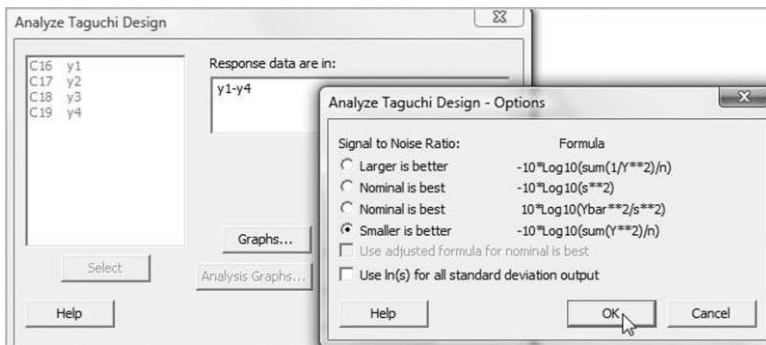

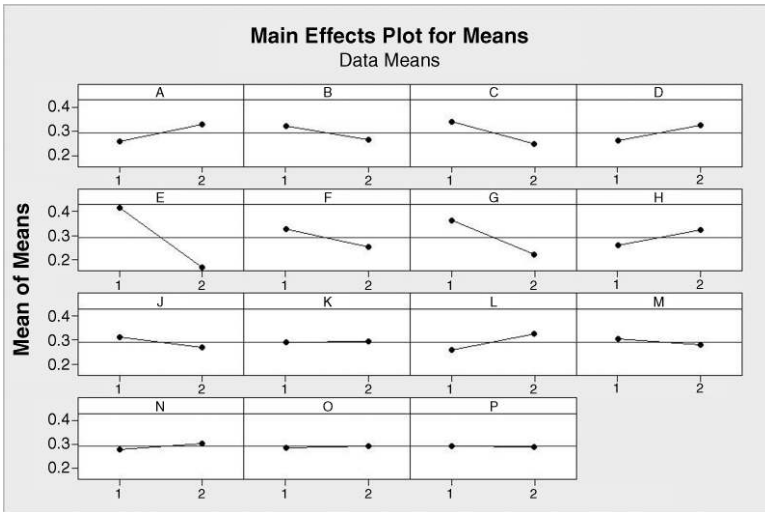**Figure 8.68**   Dialog for analysis of a Taguchi design experiment.

Figure 8.69   Main effects plot for shrinkage.

plots indicate that level 2 (existing) should be used for factor G (wire diameter). Factors such as O and P appear to have no impact on shrinkage.

Quinlan *et al.* used ANOVA on the signal to noise ratios and concluded that the eight factors E, G, K, A, C, F, D and H had significant effects. Table 8.28 gives a summary of data for 100 casings produced with the existing choice of factor levels prior to the experimentation and for 100 casings produced with the choice of factor levels indicated by the experiment as being optimal.
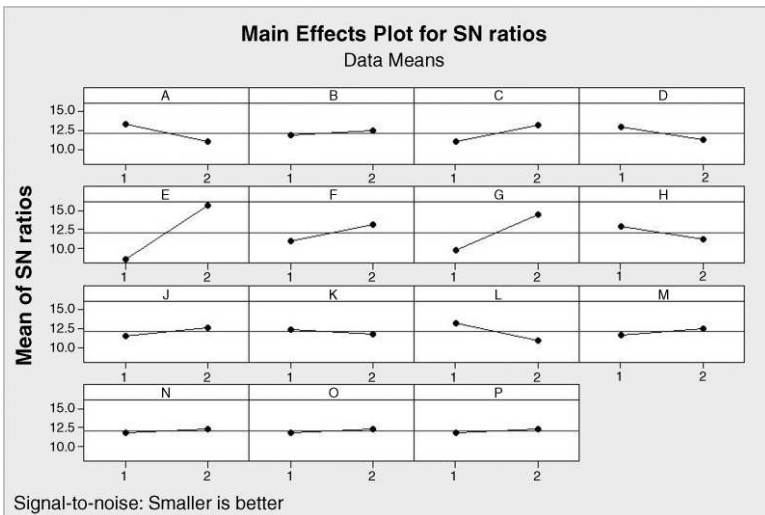

Figure 8.70   Main effects plot for signal to noise ratio.

Table 8.28   Summary data for sets of 100 casings before and after process changes.

Levels	Mean	Standard deviation	Signal to noise ratio	Predicted signal to noise ratio
Existing	0.26	0.050	11.6	12.6
Optimal	0.05	0.025	25.0	24.3

Figure 8.71 displays approximate distributions of shrinkage for the two phases of production. Quinlan *et al.* commented:

> this dramatic improvement . . . was only achieved by changing one of the design criteria of the product. The control charting efforts that had been assiduously applied to this product and process could not have been successful in reducing the average post extrusion shrinkage by the amount shown.

The interested reader will find further discussion of this experiment and of Taguchi's contribution to quality improvement in Box *et al.* (1988).

In the previous example each column of the orthogonal array had a factor allocated to it. In the next example, reported by Pignatiello and Ramberg (1985, pp. 198–206), the $L_8(2^7)$ array, which may be used to investigate seven factors, was used to investigate four factors. The process that they investigated involved the manufacture of vehicle leaf springs. The spring assembly passes through a high-temperature furnace and is then transferred to a press where the curvature is induced. Finally, the assembly is quenched in oil. The process should yield springs with a free height of 8 inches. The four controllable factors were:

- *A*, furnace temperature (°F) with levels 1880 (1) and 1840 (2);
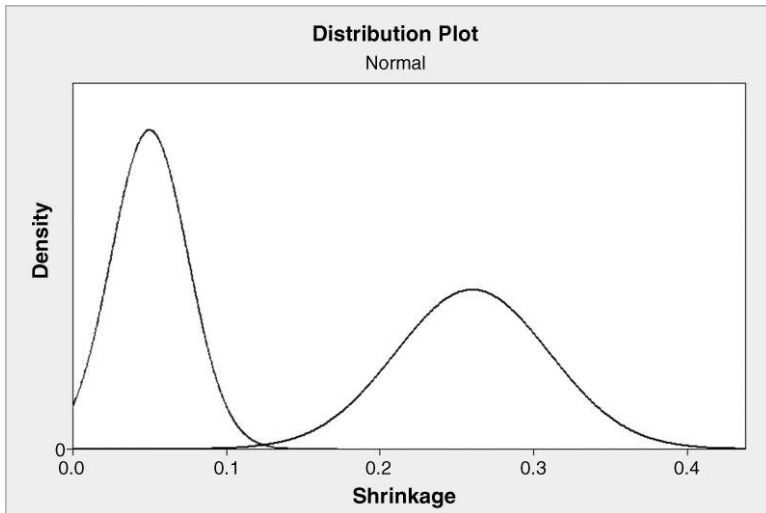
- *B*, heating time (s) with levels 23 (1) and 25 (2);

Figure 8.71   Approximate distributions of shrinkage before and after process changes.

- C, transfer time (s) with levels 10 (1) and 12 (2);

- D, hold-down time (s) with levels 3 (1) and 2 (2).

The reader should note that the experimenters did not always use the lowest value of a factor setting as level 1.

The linear graphs associated with the Taguchi $L_8(2^7)$ array are displayed in Figure 8.72 and are reproduced by permission of the American Supplier Institute Inc., Dearborn, MI.

Engineers involved with the process suspected that interactions were likely between all possible pairs of the factors A, B and C but that factor D was unlikely to interact with the other three. Thus they used Graph 1 to make the allocation of factors A, B, C and D to columns 1, 2, 4 and 7 respectively as indicated by the shading in Table 8.29. These four columns give the eight FLCs to be used in the experiment. The line segment labelled 3 in Graph 1, which joins the triangle vertices labelled 1 (allocated to factor A) and 2 (allocated to factor B), indicates that column 3 corresponds to the AB interaction. The reader is invited to verify from Graph 1 that columns 5 and 6 correspond to the AC and BC interactions respectively. Columns 3, 5 and 6 have no bearing on the actual running of the experiment but may be used in subsequent analysis of the experimental data to investigate AB, AC and BC interaction effects. This array was what is referred to in the Taguchi methodology as the inner array for the experiment.

Table 8.29   The $L_8(2^7)$ orthogonal array.

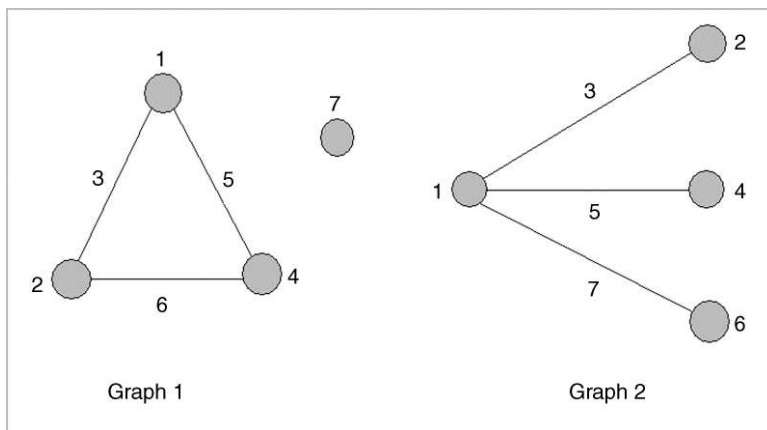| A	B	AB	C	AC	BC	D
1	1	1	1	1	1	1
1	1	1	2	2	2	2
1	2	2	1	1	2	2
1	2	2	2	2	1	1
2	1	2	1	2	1	2
2	1	2	2	1	2	1
2	2	1	1	2	2	1
2	2	1	2	1	1	2

Figure 8.72   Linear graphs for the $L_8(2^7)$ orthogonal array.

Table 8.30   Data for spring free height experiment.

Inner array		Factor			Outer array O					
	A	B	C	D	1				2	
	1	1	1	1	7.56	7.81	7.69	7.81	7.50	7.79
	1	1	2	2	7.59	7.56	7.75	7.63	7.75	7.56
	1	2	1	2	7.69	8.09	8.06	7.56	7.69	7.62
	1	2	2	1	8.15	8.18	7.88	7.88	7.88	7.44
	2	1	1	2	7.56	7.62	7.44	7.18	7.18	7.25
	2	1	2	1	7.50	7.56	7.50	7.50	7.56	7.50
	2	2	1	1	7.94	8.00	7.88	7.32	7.44	7.44
	2	2	2	2	7.78	7.78	7.81	7.50	7.25	7.12

The temperature of the oil used for quenching the springs once formed was difficult to control so it was treated as a noise factor. For each FLC, three springs were produced with factor O, oil quench temperature (°F), at level 1 (130–150°F) and three were produced with O at level 2 (150–170°F). Factor O with its two levels provides what is referred to in the Taguchi methodology as the outer array for the experiment. The design and data are given in Table 8.30.

We will now work through the design and analysis of the experiment, using Minitab. The first step is to create the design using **Stat > DOE > Taguchi > Create Taguchi Design**. On specifying 2-Level Design under **Type of Design** and **Number of factors:** 4, clicking on **Designs. . .** reveals four available designs – $L_8(2^7)$, $L_{12}(2^7)$, $L_{16}(2^7)$ and $L_{32}(2^7)$. In this case L_8 was used. **Factors. . .** may be used to name the factors and specify the levels as before, as shown in Figure 8.73.

Abbreviated names were used. Note that the default allocation of the factors A, B, C and D is to columns 1, 2, 4 and 7 which is what is required in this case. (Were the design associated
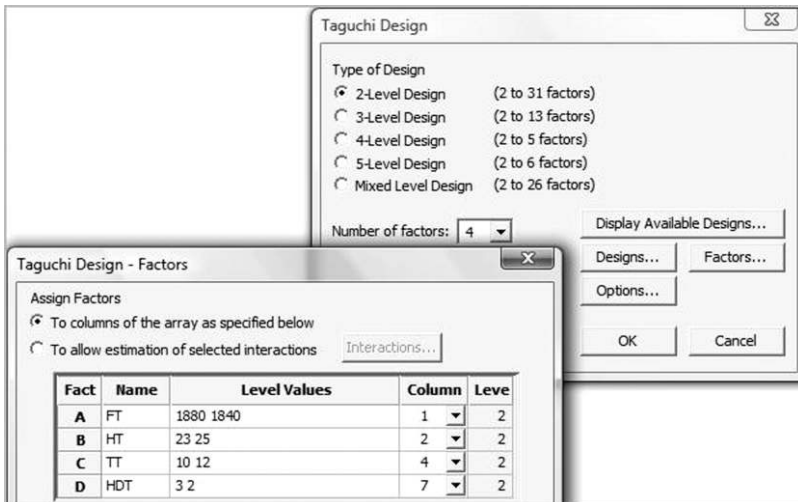

Figure 8.73   Specifying the factors and levels.

FT	HT	TT	HDT	h1 (O Low)	h2 (O Low)	h3 (O Low)	h4 (O High)	h5 (O High)	h6 (O High)
1880	23	10	3	7.56	7.81	7.69	7.81	7.50	7.79
1880	23	12	2	7.59	7.56	7.75	7.63	7.75	7.56
1880	25	10	2	7.69	8.09	8.06	7.56	7.69	7.62
1880	25	12	3	8.15	8.18	7.88	7.88	7.88	7.44
1840	23	10	2	7.56	7.62	7.44	7.18	7.18	7.25
1840	23	12	3	7.50	7.56	7.50	7.50	7.56	7.50
1840	25	10	3	7.94	8.00	7.88	7.32	7.44	7.44
1840	25	12	2	7.78	7.78	7.81	7.50	7.25	7.12

Figure 8.74   Design and data.

with the other linear graph in Figure 8.72 required, then the drop-down menu for the **Column** associated with factor D could be used to change the 7 to a 6.) The reader should note that under **Options. . .** there is no scope for randomization. The data are shown, added to the design worksheet, generated by Minitab, in Figure 8.74. This worksheet is supplied as Springs. MTW, and the data are reproduced with permission from the *Journal of Quality Technology* (© 1985 American Society for Quality).

Columns C5, C6 and C7 contain the heights of springs obtained with the eight controllable FLCs for the Taguchi design and with low oil quench temperature. Columns C8, C9 and C10 contain the heights of springs obtained with the eight controllable FLCs for the Taguchi design and with high oil quench temperature.

Central to Taguchi's methodology is the determination of levels of the controllable factors which maximize signal to noise ratio and then using factors which do not influence the signal to noise ratio to adjust the actual response, i.e. in this case the spring free height, to the desired target value.

The analysis may be performed using **Stat** > **DOE** > **Taguchi** > **Analyze Taguchi Design. . .**. Under **Response data are in:** the six columns containing the data were selected. The nominal is best signal to noise ratio $SN_T = 10 \log_{10}(\bar{y}^2/s^2)$ was selected under **Options. . .**. Under **Graphs. . .** the options to **Generate plots of main effects and interactions in the model for** both signal to noise ratios and means were checked. For the **Interaction plots** only **Display interaction plot matrix** was checked. **Terms . . .** was used to select the four main effects of interest (A, B, C and D) plus the three two-factor (first-order) interactions of interest (AB, AC and BC). Under **Analysis. . . Signal to Noise ratios** and **Means** were checked, both under **Display response tables for** and under **Fit linear model for**. No use was made of **Analysis Graphs. . .** or **Storage. . .**.

The main effects plot for the signal to noise ratios is displayed in Figure 8.75. The most important effect appears to be that of B, heating time. Bearing in mind that signal to noise ratios were developed in such a way that the higher the value the better the indication is that a heating time of 23 seconds appears advisable.

The interaction plot matrix for the signal to noise ratios is shown in Figure 8.76. The experimenters were led to believe that the interaction between heating time and transfer time (BC) was most important. Note that the greatest mean signal to noise ratio is associated with the FLC of heating time 23 seconds and transfer time 12 seconds. The experimenters used ANOVA to provide formal evidence that the main effect of heating time and the interaction effect between heating time and transfer time effects were 'active'.

The main effects plot for means of the spring heights is displayed in Figure 8.77. This plot suggested to the experimenters that furnace temperature and hold-down time could be used to adjust spring height to the target value of 8 inches. Thus the levels for furnace temperature of 1880°F and for hold-down time of 3 seconds are indicated.
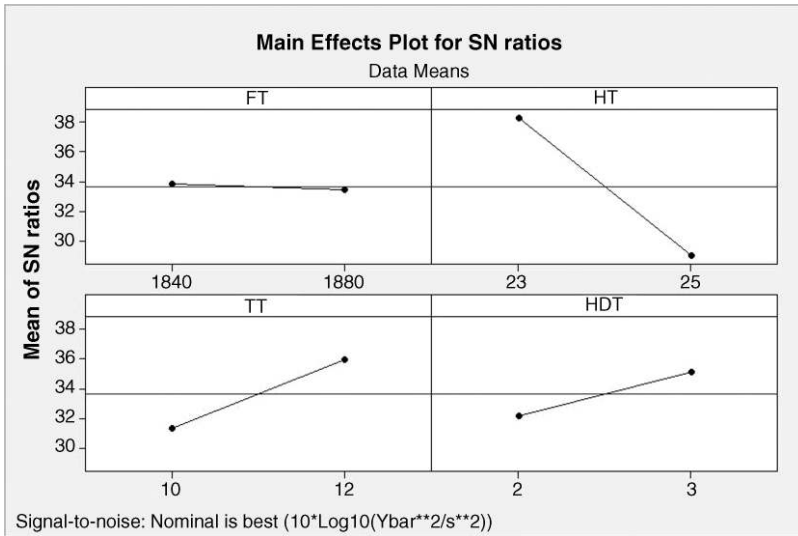
Figure 8.75   Main effects plot for signal to noise ratios.

Finally, the experimenters determined that the mean free height obtained with low oil quench temperatures was 7.77 inches and that the mean free height obtained with high oil quench temperatures was 7.51 inches. This led to the decision to attempt to control the oil quench temperature in the range 130–150°F.

Changes made to the process operating standards on the basis of the experimental findings led to a 60% reduction in the process variability and to a reduction in the deviation of mean free height from target from 0.026 inches to 0.014 inches on average. In conclusion the authors
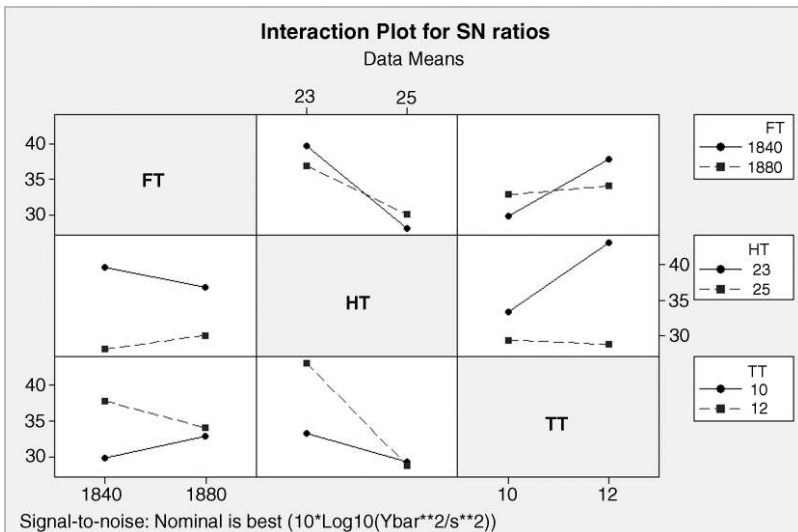

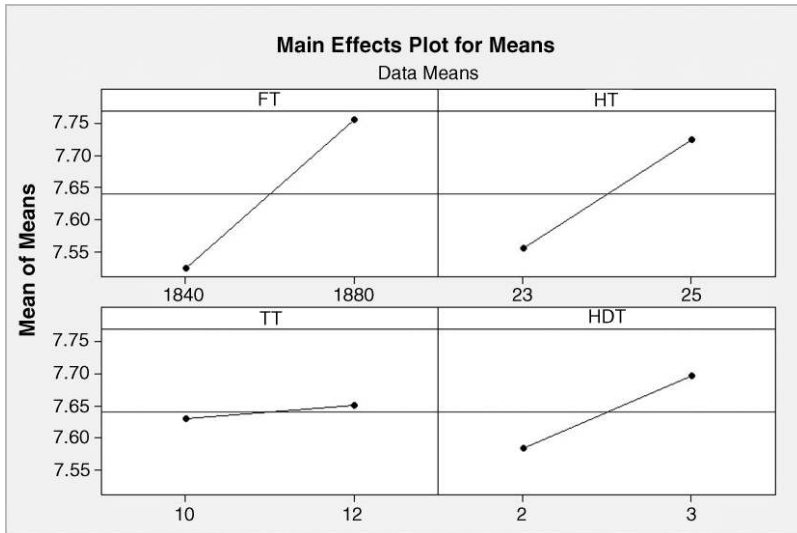**Figure 8.76**   Interaction plots for signal to noise ratios.

Figure 8.77 Main effects plot for means.

commented: 'Engineers without extensive background in experimental design were able to use the Taguchi method to guide them through a major reduction in process variability and an assessment of control factors'.

Hicks and Turner (1999, p. 398) make a number of observations on the use of Taguchi experimental designs. Three are quoted below:

- 'Taguchi's arrays, when used on the maximum number of factors, are all of resolution III. This means that all main effects are independent of other main effects but are confounded with two-way interactions. Such designs are seldom recommended because of their low resolution number.'

- 'Taguchi-type designs should be considered only as a screening type of experiment.'

- 'To really find out what factors and interactions are important, one should run either a complete factorial or a fractional factorial of resolution V or higher.'

8.5   Exercises and follow-up activities

1. Chemical process engineers conducted a designed experiment in order to investigate the influence of the factors cure temperature (°C) and amount of filler (kg) on the response hardness of a rubber compound. The levels of temperature selected were 120, 160 and 200, and the levels of amount selected were 10, 15 and 20.

    (i) Given that it was decided to have two replications, use **Stat** > **DOE** > **Factorial** > **Create Factorial Design** to create a pro forma that the experimenters could have used to conduct the experiment and to record the results. The actual design used and the results of the experiment are provided in worksheet Hardness.MTW.

(ii) Display the data in both main effects and interaction plots using **Stat** > **DOE** > **Factorial** > **Factorial Plots. . .**.

(iii) State the factor–level combination that appears best on the basis of these plots, given that high hardness is desirable.

(iv) Use **Stat** > **DOE** > **Factorial** > **Analyze Factorial Design. . .** to carry out a formal analysis, including examination of plots of the residuals.

(v) State recommendations for running the process that can be made.

2. The marketing department of a large company carried out a two-factor designed experiment in order to assess the potential impact of both the design and size of a newspaper advertisement on sales. The design had levels 1, 2 and 3 and the size had levels 1 and 2, with 2 corresponding to the larger size. The response was sales projection. Display and analyse the data, available in Adverts.MTW.

3. A 2^2 factorial experiment was used to investigate yield (g), Y, from a batch culture process for yeast. The factors of interest were the amount of sugar used (A, g1^{-1}), and air flow (B, 1h^{-1}). The levels for A were 40 and 60 and the levels for B were 2 and 3. Use **Stat** > **DOE** > **Factorial** > **Create Factorial Design** with the option **2-level factorial (default generators)** to set up a pro forma for the experiment, with two replications. Create and print two copies of a 'blank' cube plot that could be used to record and display mean yield for each factor–level combination.

 Given the data obtained in Table 8.31 for scenario 1,

(i) calculate the mean yield for each factor–level combination and add the means to one of your blank cube plots;

(ii) with the aid of your cube plot, calculate the main effects of sugar and air flow and their interaction effect.

 Given further that the design and data are available in Yeast1.MTW,

(iii) use Minitab to display main effects and interaction plots;

(iv) use **Stat** > **DOE** > **Factorial** > **Analyze Factorial Design. . .** to carry out a formal analysis and to create residual plots – check your calculated effects against those in the Session window output and state which effects differ significantly from zero at the 5% level of significance;

Table 8.31   Data for yield – scenario 1.

	Yield	Air flow			
		2		3	
Sugar	40	9.4		4.7	
			12.2		5.6
	60	17.5		10.1	
			21.1		16.2

Table 8.32 Data for yield – scenario 2.

		Air flow			
Yield		2		3	
Sugar	40	7.5		5.2	
			10.4		9.0
	60	21.8		9.8	
			20.1		12.1

(v) create both a surface plot and a contour plot of yield versus sugar and air flow and, given that the higher the yield the better, state the nature of further experimentation that might prove informative.

Repeat the above five steps for the data in Table 8.32 for scenario 2; the design and alternative set of data are provided in Yeast2.MTW. (Note that in this exercise the run order obtained when you create the design is unlikely to match that in the supplied worksheets.)

4. Hellstrand (1989) reported on a 2^3 experiment carried out by SKF, a manufacturer of deep groove bearings. The factors were inner ring heat treatment, outer ring osculation and cage design, and the response was life (hours) of the bearings determined in an accelerated life test procedure. For each factor the levels were standard and modified, with standard representing the current level for production of the type of bearing. The data are displayed in Table 8.33 and are reproduced by permission of The Royal Society, London.

Use Minitab to display the data in the form of a cube plot and to create an interaction plot for the heat–osculation interaction effect. (In setting up the design using Minitab, uncheck **Randomize Runs** under **Options. . .**, so that the factor–level combinations will be in standard order as in Table 8.33.) Box (1990, p. 367) commented that the experiment 'led to the extraordinary discovery that, in this particular application, the life of a bearing can be increased fivefold if the two factors outer ring osculation and inner ring heat treatments are increased together'.

5. In Exercise 8 of Chapter 3 you set up a data set on battery life based on work by Wasiloff and Hargitt (1999). Open the worksheet that you saved (or, if you did not save it, refer to Table 3.4), analyse the data and comment on the theory referred to in that exercise.

Table 8.33 Data for bearing life experiment.

Heat	Osculation	Cage	Life
Standard	Standard	Standard	17
Modified	Standard	Standard	26
Standard	Modified	Standard	25
Modified	Modified	Standard	85
Standard	Standard	Modified	19
Modified	Standard	Modified	16
Standard	Modified	Modified	21
Modified	Modified	Modified	128

6. Box *et al.* (2005, p. 236) give an example of a 2^{4-1} fractional factorial experiment with a single replication. An industrial chemist performed the experiment during the formulation of a household liquid product. The factors of interest were acid concentration, catalyst concentration, temperature and monomer concentration. The response of interest was stability, with values in excess of 25 being desirable. The data are provided in Stability.MTW and reproduced by permission of John Wiley & Sons, Inc., New York.

(i) Before opening the worksheet containing the data, create a 2^{4-1} design with a single replication using all the defaults and observe the output in Panel 8.21 that appears in the Session window. Observe that the design has resolution IV and that first-order (two-factor) interactions are confounded in pairs.

(ii) On opening Stability.MTW, the design involved in the experiment may be checked using the Show Design icon [icon]. On clicking the icon the information in Figure 8.78 is displayed. The final line lists the terms as A, B, C, D, AB, AC and AD. This indicates that, if it is assumed that the three-factor interactions and the four-factor interaction are negligible, the experiment will provide estimates of the four main effects. The alias structure displayed in Panel 8.21 reveals that the estimate corresponding to the term AB will reflect both the AB and CD interactions if neither is negligible.

(iii) Verify from a normal probability plot of the effects obtained via **Graphs. . .** under **Stat > DOE > Factorial > Analyze Factorial Design. . .** that only the main effects of acid concentration and catalyst concentration are significant at the 10% level. (Note that the default list of terms, displayed under **Terms. . .**, is A, B, C, D, AB, AC and AD, as specified in the Show Design information in Figure 8.78.)

(iv) The experiment may now be regarded as a 2^2 factorial in the factors A and B, with two replications. Reanalyse the data using **Analyze Factorial Design. . .** with the **Terms. . .** list changed to A, B and AB. There are now sufficient degrees of

```
Fractional Factorial Design

Factors:  4   Base Design:        4, 8   Resolution:   IV
Runs:     8   Replicates:         1      Fraction:     1/2
Blocks:   1   Center pts (total): 0


Design Generators: D = ABC


Alias Structure

I + ABCD

A + BCD
B + ACD
C + ABD
D + ABC
AB + CD
AC + BD
AD + BC
```

Panel 8.21   Design details for stability experiment.

```
Factors:     4    Base Design:        4, 8
Runs:        8    Replicates:            1
Blocks:    none   Center pts (total):    0

Display Order:  Run Order
Display Units:  Uncoded


Factors and Their Uncoded Levels

Factor  Name            Low   High
A       Acid             20     30
B       Catalyst          1      2
C       Temperature     100    150
D       Monomer          25     50


Responses and Models

Response:   Stability
Terms:      A B C D AB AC AD
```

Figure 8.78   Details of the experimental design.

freedom to enable *P*-values to be computed, and you should find the information in Panel 8.22 displayed in the Session window.

Thus we have formal evidence, at the 1% and 5% levels of significance respectively, of the influence of both acid and catalyst concentrations on stability. Note that clicking the Show Design icon now provides the same information as in Figure 8.78 except that list of terms has been revised in view of the changes made to the fitted model.

(v) Use **Stat > DOE > Factorial > Contour/Surface Plots. . .** to confirm the plot in Figure 8.79.

(vi) State recommendations for further experimentation.

The authors commented that 'although none of the tested conditions produced the desired level of stability, the experiments did show the direction in which such conditions might be found. A few exploratory runs performed in this direction produced for the first time a product with stability greater than the goal of 25.'

```
Estimated Effects and Coefficients for Stability (coded
units)

Term            Effect    Coef  SE Coef      T      P
Constant                14.625   0.4146  35.28  0.000
Acid            -5.750  -2.875   0.4146  -6.93  0.002
Catalyst        -3.750  -1.875   0.4146  -4.52  0.011
Acid*Catalyst    0.250   0.125   0.4146   0.30  0.778
```

Panel 8.22   Output from analysis of Stability experiment.
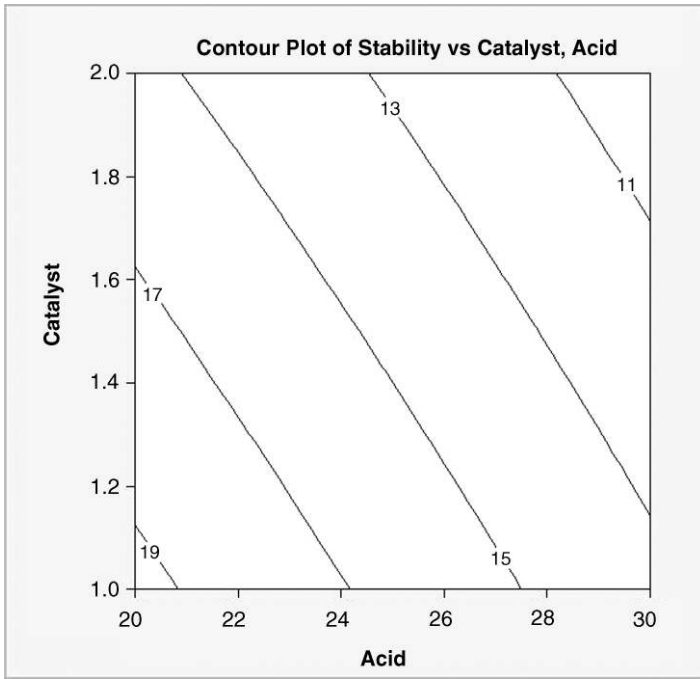
Figure 8.79   Contour plot of Stability versus Acid and Catalyst concentrations.

7. Designing, executing and analysing a factorial experiment provides valuable learning opportunities. One of the most widely used sources of data for such experiments are helicopters made according to the template in Figure 8.80.
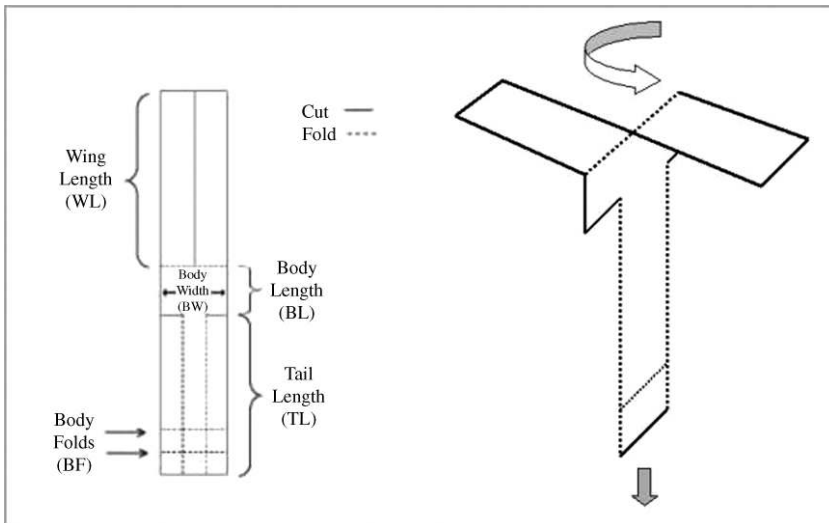

Figure 8.80   Helicopter template and sketch of helicopter in flight.

Table 8.34 Factors for helicopter design.

Factor	Level for prototype (mm)
Wing length WL	75
Body width BW	30
Body length BL	25
Tail length TL	75
Body folds BF	10

As a prototype it is suggested that a helicopter be constructed from paper with density 80 g/m^2 with the dimensions given in Table 8.34. The response of interest is the flight time (seconds) when the helicopter is released from a fixed height and allowed to descend to the floor. Flight time can be measured using an electronic stopwatch. The project team should make several flights with the prototype in order to establish standard operating procedures for launching and timing.

Given that the longer the flight time the better, the project team should then design, execute and analyse a screening experiment in the form of a fractional factorial, of at least resolution IV, that involves at least five factors and at least two replications. (Not all factors in Table 8.34 need be used. For example, paper density, with levels 80 g/m^2 and 100 g/m^2, could be used as a factor; the presence or absence of adhesive tape to maintain right angles between the wings and the body could be used as a factor.) Finally, recommendations for further experimentation should be made. Interested readers will find useful further information on helicopter experiments and on the use of experimental design to improve products and services in Box (1999) and Box and Liu (1999).

8. Two former students of the author at Edinburgh Napier University carried out a designed experiment as part of their course on experimental design taught by Dr Jeff Dodgson. The factors and levels considered are listed in Table 8.35. The design and data are available in Golf.MTW and in Dodgson (2003) and are reproduced with permission from the *Journal of Quality Technology* (© 2003 American Society for Quality).

(i) State the resolution of the design.

(ii) Demonstrate that there is evidence that the effects A, D, E and AE are significant.

(iii) Give an interpretation of the significant main effect of D.

(iv) Give an interpretation of the significant AE interaction effect.

Table 8.35 Factors and levels for golf experiment.

Factor	Level	
	-1	$+1$
A Ability (handicap)	8	4
B Ball type	Balata	Two piece
C Club type	Wood	Metal
D Ground condition	Soft	Hard
E Teeing	No tee	Tee

Table 8.36 Factors and levels for paint experiment.

Factor	Description	Levels	
		$-$	$+$
A	Component I	0.01	0.03
B	Component II	0.003	0.015
C	Component III	0.008	0.016
D	Component IV	0.01	0.02
E	Component V	0.57	0.77
F	Component VI	0.20	0.40
G	Substratum	Rigid	Flexible
H	Primer	Without	With
I	Oven Temperature	110 °C	130 °C
J	Time in Oven	20 min	30 min

9. Teresa López-Alvarez and Aguirre-Torres (1997) employed a fractional factorial experiment in the development of a new paint product for use in the automotive industry. The experiment was a 2^{10-5} design of resolution IV. The factors and levels are listed in Table 8.36.

   The response of interest was a measure of yellowing of the paint on a car bumper component that had been subjected to an accelerated ageing process. The data for this example, available in the file Yellow.MTW, are reproduced with permission from *Quality Engineering* (© 1997 American Society for Quality).

   Use Minitab to confirm the conclusions reached by the experimenters:

   - 'Flexible substratum promotes the appearance of yellowing.'

   - 'Control of factors C, D and F provides the largest opportunities for improvement.'

   - 'Increasing factor E does not result in any significant improvement.'

   The authors report that the conclusion that factor E did not have a significant effect was a surprise to research personnel. Make suggestions for further experimentation.

10. It is possible to incorporate blocking into factorial and fractional factorial experiments. Consider, for example, a full 2^3 factorial with factors A, B and C in standard order as displayed in Table 8.37.

    Imagine that the planned experiment was to involve a single replication and required the use of a raw material supplied in batches. Given that a batch of raw material is sufficient for four experimental runs of the process and that it is known that there is significant variation between batches, one possibility would be to carry out the first four runs with one batch of raw material and the remaining four with a second batch, as indicated by the final column in Table 8.37. However, scrutiny of the final two columns of Table 8.37 indicates that any block effect would be confounded with the main effect of factor C.

    The correct procedure, for the design of a full 2^3 factorial experiment in two blocks, is based on the knowledge that three-factor (second-order) interactions are often

Table 8.37   Naïve blocking of a full 2^3 experiment.

A	B	C	Batch
− 1	− 1	− 1	1
1	− 1	− 1	1
− 1	1	− 1	1
1	1	− 1	1
− 1	− 1	1	2
1	− 1	1	2
− 1	1	1	2
1	1	1	2

negligible. The reader is invited to complete the calculation of the *ABC* column in Table 8.38 and the allocation of factor–level combinations for which *ABC* is –1 to batch 1 and those for which ABC is 1 to batch 2. The design is shown in standard order in Table 8.39.

Finally, the reader is invited to use **Stat > DOE > Factorial > Create Factorial Design. . .** and to select, under **Design. . .**, the full 2^3 design with **Number of center points:** 0, **Number of replicates:** 1 and **Number of blocks:** 2. The author obtained the design in Table 8.40 (randomization was used).

If one thinks of the two batches of available raw material labelled 1 and 2 then the first four experimental runs would be carried out using the second batch – think of tossing a coin

Table 8.38   Blocking of a full 2^3 experiment.

A	B	C	ABC	Batch
− 1	− 1	− 1	− 1	1
1	− 1	− 1	1	2
− 1	1	− 1	1	2
1	1	− 1	− 1	1
− 1	− 1	1		
1	− 1	1		
− 1	1	1		
1	1	1		

Table 8.39   Blocking of a full 2^3 experiment – standard order.

Standard Order	A	B	C	Batch
1	− 1	− 1	− 1	1
2	1	1	− 1	1
3	1	− 1	1	1
4	− 1	1	1	1
5	1	− 1	− 1	2
6	− 1	1	− 1	2
7	− 1	− 1	1	2
8	1	1	1	2

Table 8.40 Blocking of a full 2^3 experiment with randomization.

StdOrder	RunOrder	CenterPt	Blocks	A	B	C
6	1	1	2	-1	1	-1
5	2	1	2	1	-1	-1
7	3	1	2	-1	-1	1
8	4	1	2	1	1	1
4	5	1	1	-1	1	1
1	6	1	1	-1	-1	-1
2	7	1	1	1	1	-1
3	8	1	1	1	-1	1

to decide which batch is to be used first. Table 8.40 indicates that the corresponding factor–level combinations in standard order are numbers 5,6,7 and 8. The randomization implemented by Minitab dictates that these combinations should be run in the order 6, 5, 7 and 8 in the experiment. Similarly, with the first batch of raw material the factor–level combinations 1, 2, 3 and 4 in standard order should be run in the order 4, 1, 2 and 3.

11. An experiment was designed during the design of a chiller for pasteurized milk. The chiller consists of refrigerated plates over which the milk flows. The factors of interest were spacing of the plates, temperature of the plates and milk flow rate. The response was the score awarded to milk chilled using each factor–level combination by a panel of expert tasters. As only four experimental runs could be carried out with one tanker of milk, it was decided to run a single replicate of a full 2^3 factorial experiment in two blocks. The data are available in Milk.MTW. Create a normal probability of the effects and interpret it.

12. John Douglass and Shirley Coleman presented a case study of the application of a Taguchi experimental design at *Industrial Statistics in Action 2000* (Douglass and Coleman, 2000, pp. 11–18). The designed experiment was employed in a quality improvement project that had the aim of improving the yield at final test of breathing apparatus demand valves manufactured by Draeger Safety UK Ltd., Northumberland, UK.

    One response of interest was static pressure and the factors (and levels) were as follows:

    1. demand valve type (standard/US version);

    2. lever spring tension (standard/modified);

    3. seal diameter (9.51 mm/9.63 mm);

    4. bore diameter (9.16 mm/9.18 mm);

    5. lever height setting (low/high);

    6. diaphragm resistance (low/high).

    The specification tolerance for static pressure was 2.7 to 3.5 bar and at times reject levels approached 20% for static pressure, with the 'great majority of these rejects exceeding the upper specification limit'.

The design used was the Taguchi $L_{32}(2^{31})$ orthogonal array with five replications. The data are available in the Excel spreadsheet Valves.xls and are reproduced by permission of Draeger Safety UK Ltd. On opening the spreadsheet as a Minitab worksheet and selecting **Stat > DOE > Taguchi > Analyze Taguchi Design. . .**, a message is displayed requesting information in order to enable the software to perform the analysis. On clicking on **Yes**, a **Define Custom Taguchi Design** dialog box appears in which the six factors must then be selected. On clicking on **OK** one can then proceed to carry out the analysis.

(i) By referring to the mean response, explain why it would be desirable to operate the process using modified tension in the lever springs and high lever height setting.

(ii) With Spring and Lever as **Terms** use **Stat > DOE > Taguchi > Predict Taguchi Results. . .** to verify that a mean static pressure of 3.01 is predicted. (Under **Levels**, check Select levels from a list and specify level 2 for each of the factors of interest using the drop-down menus that appear.)

Following the introduction of these levels in routine production the reject rate was reduced to below 3%.